On the smallest poles of topological zeta functions

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Abstract

We study the local topological zeta function associated to a complex function that is holomorphic at the origin of $\mathbb{C}^2$ (respectively $\mathbb{C}^3$). We determine all possible poles less than $-1/2$ (respectively -1). On $\mathbb{C}^2$ our result is a generalization of the fact that the log canonical threshold is never in $]5/6, 1[$. Similar statements are true for the motivic zeta function.

1 Introduction

(1.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^n$ which satisfies $f(0) = 0$ and which is not identically zero. Let $g : V \to U \subset \mathbb{C}^n$ be an embedded resolution of a representative of $f^{-1}\{0\}$. We denote by E_i, $i \in T$, the irreducible components of $g^{-1}(f^{-1}\{0\})$, and by N_i and $\nu_i - 1$ the multiplicities of $f \circ g$ and $g^*(dx_1 \wedge \cdots \wedge dx_n)$ along E_i. The (N_i, ν_i), $i \in T$, are called the numerical data of the resolution (V, g). For $I \subset T$ denote also $E_I := \cap_{i \in I} E_i$ and $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$.

The set of germs of holomorphic functions on a neighbourhood of $0 \in \mathbb{C}^n$ will be denoted by $\mathcal{O}_n$.

(1.2) To f one associates the local topological zeta function

$$Z_f(s) = Z_{\mathrm{top},0,f}(s) := \sum_{I \subset T} \chi(\overset{\circ}{E}_I \cap g^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

Here s is a complex variable and $\chi(\cdot)$ denotes the topological Euler-Poincaré characteristic. The remarkable fact that $Z_f(s)$ does not depend on the chosen resolution was first proved in [DL1] by expressing it as a limit of Igusa's p-adic zeta functions.

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(1.3) The log canonical threshold $c_0(f)$ of f at $0 \in \mathbb{C}^n$ is by definition

$$\sup\{c \in \mathbb{Q} \mid \text{ the pair } (\mathbb{C}^n, c \operatorname{div} f) \text{ is log canonical in a neighbourhood of } 0\}.$$

We can describe it (see [Ko2, Prop 8.5]) in terms of the embedded resolution (V, g) as $c_0(f) = \min\{\nu_i/N_i \mid i \in T\}$. In particular, this minimum is independent of the chosen resolution. Consequently, $-c_0(f)$ is the largest candidate pole of $Z_f(s)$. The log canonical threshold has already been studied in various papers of Alexeev, Ein, Kollár, Kuwata, Mustaţă, Prokhorov, Reid, Shokurov and others; especially the sets

$$\mathcal{T}_n := \{c_0(f) \mid f \in \mathcal{O}_n\},$$

with $n \in \mathbb{Z}_{>0}$, are the subject of interesting conjectures.

It is natural to investigate whether more quotients $-\nu_i/N_i$, $i \in T$, yield invariants of the germ of f at 0. Of course, the whole set $\{-\nu_i/N_i \mid i \in T\}$ depends on the chosen resolution (for n=2 however one could consider such a set associated to the minimal resolution); but its subset consisting of the poles of $Z_f(s)$ is an invariant of f. Philosophically, these poles are induced by 'important' components E_i, which occur in every resolution. For $n \in \mathbb{Z}_{>0}$, we define the set $\mathcal{P}_n$ by

$$\mathcal{P}_n := \{s_0 \mid \exists f \in \mathcal{O}_n \,:\, Z_f(s) \textit{ has a pole in } s_0\}.$$

The case $n = 1$ is trivial: $\mathcal{T}_1 = \{1/i \mid i \in \mathbb{Z}_{>0}\}$ and $\mathcal{P}_1 = \{-1/i \mid i \in \mathbb{Z}_{>0}\}$.

(1.4) When $n = 2$, it is known that $\mathcal{T}_2 \cap]5/6, 1[= \emptyset$ (see [Ku1] or [Re]). Because it follows from [Ve4] that $-c_0(f)$ is a pole (and thus the largest pole) of $Z_f(s)$, the statement $\mathcal{P}_2 \cap] - 1, -5/6[= \emptyset$ would be a remarkable generalization. It is in fact not hard to prove this generalization. In this article, we will prove more:

$$\begin{aligned} \mathcal{P}_2 \cap] - \infty, -1/2[\;&=\; \{-1/2 - 1/i \mid i \in \mathbb{Z}_{>1}\} \\ &=\; \{-1, -5/6, -3/4, -7/10, \ldots\}. \end{aligned} \tag{1}$$

(1.5) Kollár proved in [Ko1] that $\mathcal{T}_3 \cap]41/42, 1[= \emptyset$. It turns out that there is no analogous result for $\mathcal{P}_3$. Actually, we will give examples of zeta functions with poles in $] - 1, -41/42[$ which are moreover arbitrarily near to -1. On the other hand, we prove the analogue of (1), which appears to be

$$\mathcal{P}_3 \cap] - \infty, -1[\;=\; \{-1 - 1/i \mid i \in \mathbb{Z}_{>1}\}. \tag{2}$$

In general, we expect that $\mathcal{P}_n \cap] - \infty, -(n-1)/2[= \{-(n-1)/2 - 1/i \mid i \in \mathbb{Z}_{>1}\}$.

Remark. One can easily show that $\mathcal{P}_n \cap] - \infty, -n + 1[= \emptyset$ if $n \geq 2$.

2 Curves

(2.1) We will determine $\mathcal{P}_2 \cap]-\infty, -1/2[$. Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^2$ which satisfies $f(0) = 0$ and which is not identically zero. Let (V, g) be the *minimal* embedded resolution of $f^{-1}\{0\}$. Write $g = g_1 \circ \cdots \circ g_t$ as a composition of blowing-ups g_i, $i \in T_e := \{1, \ldots, t\}$. The exceptional curve of g_i and also the strict transforms of this curve are denoted by E_i. The irreducible components of $f^{-1}\{0\}$ and their strict transforms are denoted by E_j, $j \in T_s$.

(2.2) The dual (minimal) embedded resolution graph of $f^{-1}\{0\}$ is obtained as follows. One associates a vertex to each exceptional curve in the minimal embedded resolution (represented by a dot), and to each branch of the strict transform of $f^{-1}\{0\}$ (represented by a circle). One also associates to each intersection an edge, connecting the corresponding vertices. The fact that E_i has numerical data (N_i, ν_i) is denoted by $E_i(N_i, \nu_i)$.

(2.3) Let E_i be an exceptional curve and let E_j, $j \in J$, be the components that intersect E_i in V. Set $\alpha_j = \nu_j - (\nu_i/N_i)N_j$ for $j \in J$. Then we have the relation

$$\sum_{j \in J} (\alpha_j - 1) + 2 = 0, \tag{3}$$

which was first proved by Loeser in [Lo], and later more conceptually by the second author in [Ve1].
Suppose that $\alpha_j \neq 0$, which is equivalent to $-\nu_i/N_i \neq -\nu_j/N_j$, for all $j \in J$. Then the contribution of E_i to the residue $\mathcal{R}$ of $Z_f(s)$ at the candidate pole $-\nu_i/N_i$ is

$$\frac{1}{N_i}\left(\chi(\overset{\circ}{E}_{\{i\}}) + \sum_{j \in J} \alpha_j^{-1}\right). \tag{4}$$

From (3) and (4) it follows that $\mathcal{R} = 0$ if J contains one or two elements. This is the easy part of the following theorem. The other part is more difficult and is proved in [Ve4].

(2.4) Theorem. *We have that s_0 is a pole of $Z_f(s)$ if and only if $s_0 = -\nu_i/N_i$ for some exceptional curve E_i intersecting at least three times other components, or $s_0 = -1/N_j$ for some irreducible component E_j of the strict transform of $f^{-1}\{0\}$.*

The following lemma is obtained by elementary calculations.

(2.5) Lemma. *Suppose that we have blown up k times but we have not yet an embedded resolution. Let P be a point of the strict transform of $f^{-1}\{0\}$ with* *multiplicity μ in which we do not have normal crossings yet. Let g_{k+1} be the blowing-up at P.*

(a) Suppose that two exceptional curves E_i and E_j contain P. Then the new candidate pole $-\nu_{k+1}/N_{k+1} = -(\nu_i + \nu_j)/(N_i + N_j + \mu)$ is larger than $\min\{-\nu_i/N_i, -\nu_j/N_j\}$.

(b) Suppose that exactly one exceptional curve E_i contains P and that $\mu \geq 2$. Then E_{k+1} has numerical data $(N_i+\mu, \nu_i+1)$ and $-(\nu_i+1)/(N_i+\mu)$ is in between $-1/\mu$ and $-\nu_i/N_i$.

(c) Suppose that exactly one exceptional curve E_i contains P and that $\mu = 1$. Remark that the two curves are tangent at P because we do not have normal crossings at P. Let g_{k+2} be the blowing-up at $E_i \cap E_{k+1}$. Because the strict transform of $f^{-1}\{0\}$ does not intersect E_{k+1} after this blowing-up, we do not have to blow up at a point of E_{k+1} anymore. Because E_{k+1} is intersected once, it follows from (2.3) that the contribution of E_{k+1} to the residue at the candidate pole $-\nu_{k+1}/N_{k+1}$ is zero. The numerical data of E_{k+2} are $(2N_i + 2, 2\nu_i + 1)$, and $-(2\nu_i + 1)/(2N_i + 2)$ is in between $-1/2$ and $-\nu_i/N_i$.

(2.6) Suppose that after some blowing-ups, we do not have normal crossings at a point P. Suppose also that the candidate poles associated to the exceptional curves through P are all larger than or equal to $-1/2$. Then it follows from the above lemma that the components above P in the final resolution do not give a contribution to a pole less than $-1/2$.

Corollary. *Zeta functions of singularities of multiplicity at least 4 do not have a pole in $]-\infty, -1/2[\setminus\{-1\}$.*

Indeed, every exceptional curve in the minimal resolution of $f^{-1}\{0\}$ lies above a point of E_1 (considered in the stage when it is created), which has a candidate pole larger than or equal to $-1/2$.

(2.7) To deal with multiplicity 2 and 3, we will study an 'easier' element of $\mathcal{O}_2$ with the same zeta function. We mention two methods to obtain a simple function with this property.

METHOD 1. (See [Ku2]) Let $f \in \mathcal{O}_n$ have multiplicity d and let f_d be the homogeneous part of degree d in the Taylor series of f. Let $N \in \mathbb{Z}_{>d}$. Take a maximal set $\mathcal{V}$ of homogeneous polynomials of degree larger than d and at most N which are linearly independent in the quotient vector space $\mathcal{O}_n/(\partial f_d/\partial x_1, \ldots, \partial f_d/\partial x_n)$. Then f is holomorphically equivalent to $f_d + \sum_{u_i \in \mathcal{V}} a_i u_i + \varphi$, for some $a_i \in \mathbb{C}$ and some $\varphi \in \mathcal{O}_n$ which satisfies $\mathrm{mult}(\varphi) > N$.

Remark. (i) The similar statement in [Ku2, Lemma 3.2] is not correct; some homogeneity condition is needed.

(ii) Let $f^{-1}\{0\}$ have an isolated singularity at the origin and suppose that we have an embedded resolution which is an isomorphism outside this singularity.

Then φ does not influence the embedded resolution and the numerical data if N is big enough. Consequently, to calculate the local topological zeta function, we can omit φ if we take N big enough. Note that when $n = 2$, $f^{-1}\{0\}$ has an isolated singularity at 0 if and only if f is reduced and $\mathrm{mult}(f) \geq 2$.

METHOD 2. *(Weierstrass Preparation Theorem)* If $f(z_1, \ldots, z_{n-1}, w) = f(z, w) \in \mathcal{O}_n$ is not identically zero on the w-axis, then f can be written uniquely as $f = (w^e + a_1(z)w^{e-1} + \cdots + a_e(z))h$, where $a_i(z) \in \mathcal{O}_{n-1}$ satisfies $a_i(0) = 0$ and $h \in \mathcal{O}_n$ satisfies $h(0) \neq 0$.
Because $h(0) \neq 0$, the resolutions and the local topological zeta functions of f and $w^e + a_1(z)w^{e-1} + \cdots + a_e(z)$ are the same. After an appropriate coordinate transformation, the desired form will appear. For example, the coordinate transformation $(z, w) \mapsto (z, w - a_1(z)/e)$ cancels the term $a_1(z)w^{e-1}$.

(2.8) *Example.* Let $f \in \mathcal{O}_2$ have multiplicity 3 and let $f_3 = y^3 + xy^2 = y^2(y + x)$. First we illustrate method 1. Because $\partial f_3/\partial x = y^2$ and $\partial f_3/\partial y = 3y^2 + 2xy$, we get $(\partial f_3/\partial x, \partial f_3/\partial y) = (y^2, xy)$. Therefore, we set $\mathcal{V} = \{x^4, x^5, \ldots, x^N\}$, and we obtain that f is holomorphically equivalent to a function of the form $y^3 + xy^2 + a_4x^4 + \cdots + a_Nx^N + g(x, y)$, with $\mathrm{mult}(g(x, y)) > N$. If all a_i can be taken zero for every N, then f is holomorphically equivalent to $y^3 + xy^2$. If there exists an a_i different from zero, which is the reduced case, the form above will allow us to calculate the local topological zeta function of f.

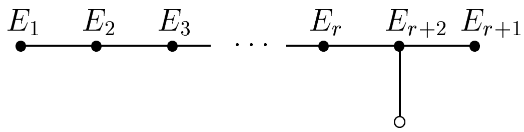
Now we illustrate method 2. By the Weierstrass preparation theorem, we may work with a function of the form $y^3 + a_1(x)y^2 + a_2(x)y + a_3(x)$, with $\mathrm{mult}(a_1(x)) = 1$, $\mathrm{mult}(a_2(x)) \geq 3$ and $\mathrm{mult}(a_3(x)) \geq 4$. One can check that there exists a coordinate transformation $(x, y) \mapsto (x, y - k(x))$ such that the function becomes of the form $y^3 + b_1(x)y^2 + b_3(x)$, with $\mathrm{mult}(b_1(x)) = 1$ and $\mathrm{mult}(b_3(x)) \geq 4$. After another coordinate transformation, we get the form $y^3 + xy^2 + g(x)$, with $\mathrm{mult}(g(x)) \geq 4$.

(2.9) Theorem. *We have*

$$\mathcal{P}_2 \cap \left]-\infty, -\frac{1}{2}\right[= \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}$$

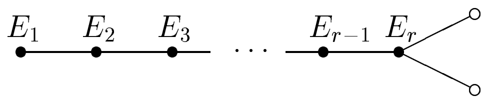
and every local topological zeta function has at most one pole in $]-1, -1/2]$.

Proof. (a) Suppose that $\mathrm{mult}(f)$, the multiplicity of f at the origin of $\mathbb{C}^2$, is equal to 2. Then f is holomorphically equivalent to $y^2 + x^k$ for some $k \in \mathbb{Z}_{>1} \cup \{0\}$. If $k = 0$, the only pole of $Z_f(s)$ is $-1/2$. If $k = 2$, the only pole of $Z_f(s)$ is -1. If k is odd, write $k = 2r + 1$. After r blowing-ups, the strict transform of $f^{-1}\{0\}$ is nonsingular and tangent to E_r. The numerical data of E_i, $i = 1, \ldots, r$, are $(2i, i + 1)$. To get the minimal embedded resolution, we now blow up twice. The dual resolution graph and the numerical data are given below.



$E_1(2,2)$ $\quad E_r(2r, r+1)$
$E_2(4,3)$ $\quad E_{r+1}(2r+1, r+2)$
$E_3(6,4)$ $\quad E_{r+2}(4r+2, 2r+3)$

If k is even and larger than 2, write $k = 2r$. Easy calculations give the following dual resolution graph.



$E_1(2,2)$
$E_2(4,3)$ $\quad E_{r-1}(2r-2, r)$
$E_3(6,4)$ $\quad E_r(2r, r+1)$

Because $-(2r+3)/(4r+2) = -1/2-1/(2r+1)$ and $-(r+1)/(2r) = -1/2-1/(2r)$, it follows from (2.4) that

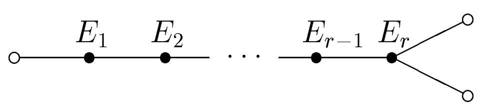
$$\begin{aligned} \{s_0 \mid \exists f \in \mathcal{O}_2 \quad &: \quad \mathrm{mult}(f) = 2 \text{ and } Z_f(s) \text{ has a pole in } s_0\} \\ &= \quad \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\} \cup \left\{ -\frac{1}{2} \right\}. \end{aligned}$$

Remark that Newton polyhedra could also be used to deal with (a), see [DL1].

(b) Suppose that $\mathrm{mult}(f) = 3$. Up to an affine coordinate transformation, there are three cases for f_3.

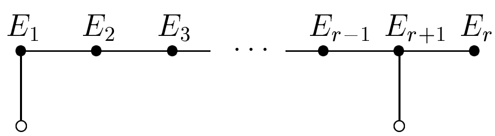
(b.1) Case $f_3 = xy(x+y)$. After one blowing-up we get an embedded resolution. The poles of $Z_f(s)$ are -1 and $-2/3 = -1/2 - 1/6$.

(b.2) Case $f_3 = y^2(y+x)$. According to example 2.8, we may suppose that $f = y^3 + xy^2 + g(x)$, where $g(x)$ is a holomorphic function in the variable x of multiplicity $k \geq 4$. If $g(x) = 0$, the poles of $Z_f(s)$ are -1 and $-1/2$. Consider now the case that k is odd. Write $k = 2r+1$. After r blowing-ups we get an embedded resolution with the following dual resolution graph and numerical data.



$E_1(3,2)$ $\quad E_{r-1}(2r-1, r)$
$E_2(5,3)$ $\quad E_r(2r+1, r+1)$

If k is even, write $k = 2r$. After $r+1$ blowing-ups we get the following picture.



$E_1(3,2)$ $\quad E_{r-1}(2r-1, r)$
$E_2(5,3)$ $\quad E_r(2r, r+1)$
$E_3(7,4)$ $\quad E_{r+1}(4r, 2r+1)$

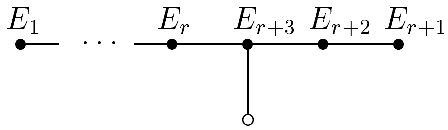
The poles appearing in (b.2) are in the desired set because $-(r+1)/(2r+1) = -1/2 - 1/(4r+2)$ and $-(2r+1)/(4r) = -1/2 - 1/(4r)$.

(b.3) Case $f_3 = y^3$. We may suppose that f is of the form

$$y^3 + a_4x^4 + b_3yx^3 + a_5x^5 + b_4yx^4 + a_6x^6 + b_5yx^5 + \cdots,$$

where $a_i, b_i \in \mathbb{C}$. If $f = f_3 = y^3$ then the only pole of $Z_f(s)$ is $-1/3$. Otherwise there is an integer $r \geq 1$ such that after blowing up r times and always taking the charts determined by $g_i(x,y) = (x, xy)$, we get $(g_1 \circ \cdots \circ g_r)^* dx \wedge dy = x^r dx \wedge dy$ and $f \circ g_1 \circ \cdots \circ g_r = x^{3r}(y^3 + a_{3r+1}x + b_{2r+1}yx + a_{3r+2}x^2 + b_{2r+2}yx^2 + a_{3r+3}x^3 + \cdots)$, with $a_{3r+1}, b_{2r+1}, a_{3r+2}, b_{2r+2}$ and a_{3r+3} not all zero. The equation of E_r in this chart is $x = 0$ and the numerical data of E_r are $(3r, r+1)$. The zero locus of $y^3 + a_{3r+1}x + b_{2r+1}yx + a_{3r+2}x^2 + b_{2r+2}yx^2 + a_{3r+3}x^3 + \cdots$ is the strict transform of $f^{-1}\{0\}$. Remark that it intersects only E_r at this stage.

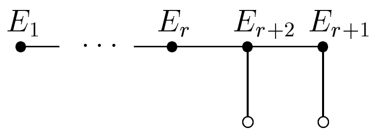
(b.3.i) If $a_{3r+1} \neq 0$, we obtain the following after blowing up three more times.



$E_r(3r, r+1)$ $\quad$ $E_{r+2}(6r+2, 2r+3)$
$E_{r+1}(3r+1, r+2)$ $\quad$ $E_{r+3}(9r+3, 3r+4)$

The pole $-(3r+4)/(9r+3)$ is in the interval $]-\infty, -1/2]$ if and only if $r = 1$, and in this case the pole is equal to $-1/2 - 1/12$.

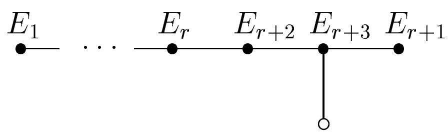
(b.3.ii) If $a_{3r+1} = 0$ and $b_{2r+1} \neq 0$, calculations give us the following data.



$E_r(3r, r+1)$ $\quad$ $E_{r+2}(6r+3, 2r+3)$
$E_{r+1}(3r+2, r+2)$

The pole $-(2r+3)/(6r+3)$ is in the interval $]-\infty, -1/2]$ if and only if $r = 1$, and in this case the pole is equal to $-1/2 - 1/18$.

(b.3.iii) If $a_{3r+1} = b_{2r+1} = 0$ and $a_{3r+2} \neq 0$, we get the following.



$E_r(3r, r+1)$ $\quad$ $E_{r+2}(6r+3, 2r+3)$
$E_{r+1}(3r+2, r+2)$ $\quad$ $E_{r+3}(9r+6, 3r+5)$

The pole $-(3r+5)/(9r+6)$ is in the interval $]-\infty, -1/2]$ if and only if $r = 1$ and in this case the pole is equal to $-1/2 - 1/30$.

(b.3.iv) The last case is $a_{3r+1} = b_{2r+1} = a_{3r+2} = 0$ and ($b_{2r+2} \neq 0$ or $a_{3r+3} \neq 0$). If $y^3 + b_{2r+2}yx^2 + a_{3r+3}x^3$ is a product of three distinct linear factors, we get an embedded resolution after one blowing-up. The numerical data of E_{r+1} are $(3r+3, r+2)$ and $-(r+2)/(3r+3) \notin]-\infty, -1/2[$.
If $y^3 + b_{2r+2}yx^2 + a_{3r+3}x^3$ is not a product of three distinct linear factors, then it is equal to $y^3 + xy^2$ after an affine coordinate transformation that does not change the equation $x = 0$ of E_r. Let g_{r+1} be the blowing-up at the origin of the chart we consider. The strict transform of $f^{-1}\{0\}$ only intersects the exceptional curve E_{r+1}, which has numerical data $(3r+3, r+2)$. Because $-(r+2)/(3r+3) \geq -1/2$ for all r, it follows from (2.4) and (2.6) that $Z_f(s)$ has no pole in $]-\infty, -1/2[$ different from -1 .

(c) Suppose that $\mathrm{mult}(f) \geq 4$. We explained in (2.6) that $Z_f(s)$ has no pole in $]-\infty, -1/2[$ different from -1. $\square$

(2.10) We now present a similar result for the following generalized zeta functions [DL1]. The case $d = 2$ is used in the next section. To $f \in \mathcal{O}_n$ and $d \in \mathbb{Z}_{>0}$ one associates the local topological zeta function

$$Z_f^{(d)}(s) = Z_{\mathrm{top},0,f}^{(d)}(s) := \sum_{\substack{I \subset T \\ \forall i \in I\, :\, d | N_i}} \chi(\overset{\circ}{E}_I \cap g^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

For $n, d \in \mathbb{Z}_{>0}$, we set

$$\mathcal{P}_n^{(d)} := \{s_0 \mid \exists f \in \mathcal{O}_n \, : \, Z_f^{(d)}(s) \text{ has a pole in } s_0\}.$$

Consequently, $Z_f(s) = Z_f^{(1)}(s)$ and $\mathcal{P}_n = \mathcal{P}_n^{(1)}$.

(2.11) Let E_i be an exceptional curve and let E_j, $j \in J$, be the components that intersect E_i in V. Then

$$\sum_{j \in J} N_j \equiv 0 \pmod{N_i}, \tag{5}$$

see e.g. [Lo] or [Ve2]. Fix $d \in \mathbb{Z}_{>0}$ and suppose that $d \mid N_i$. Let $J_d \subset J$ be the subset of indices j satisfying $d \mid N_j$. Suppose that $\alpha_j := \nu_j - (\nu_i/N_i)N_j$ is different from 0 for all $j \in J_d$. Then the contribution of E_i to the residue $\mathcal{R}$ of $Z_f^{(d)}(s)$ at the candidate pole $-\nu_i/N_i$ is

$$\frac{1}{N_i}\left(\chi(\overset{\circ}{E}_{\{i\}}) + \sum_{j \in J_d} \alpha_j^{-1}\right). \tag{6}$$

This contribution is zero if J contains one or two indices. Indeed, if J contains one element, relation (5) implies that $J = J_d$. Therefore, the contribution $\mathcal{R}$ is the same as in the case $d = 1$ and by (2.3) we get $\mathcal{R} = 0$. If J contains two elements, relation (5) implies that $J_d = J$ or $J_d = \emptyset$. If $J_d = J$, we obtain $\mathcal{R} = 0$ analogously as in the previous case. If $J_d = \emptyset$, we get $\mathcal{R} = 0$ because the Euler-Poincaré characteristic of a projective line minus two points is zero.

(2.12) Theorem. *Let $f \in \mathcal{O}_2$ and $d \in \mathbb{Z}_{>1}$. Then*

$$\mathcal{P}_2^{(d)} \cap \left]-\infty, -\frac{1}{2}\right[= \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>2} \text{ and } d|\mathrm{lcm}(2,i)\right\}.$$

Proof. We use the notation and the calculations of (2.9). First we consider the candidate pole $-\nu_j/N_j$, $j \in T_s$. Because E_j has numerical data $(N_j, 1)$, the candidate pole $-\nu_j/N_j$ is less than $-1/2$ if and only if $N_j = 1$, and in this case $d \nmid N_j$.

(a) Suppose that $\mathrm{mult}(f) = 2$. If $k = 0$ and $d = 2$, the only pole of $Z_f^{(d)}(s)$ is $-1/2$. If $k = 0$ and $d > 2$ and if $k = 2$, we obtain that $Z_f^{(d)}(s)$ is identically zero.
If $k = 2r+1$, we obtain from (2.11) and the consideration above about E_j, $j \in T_s$, that only the candidate pole associated to E_{r+2} can be a pole. So we have to compute the residue at the candidate pole $-1/2 - 1/(2r+1)$. If $d \nmid N_{r+2}$, then $\mathcal{R} = 0$. If $d|N_r$, $d|N_{r+1}$ and $d|N_{r+2}$, then $d = 1$ and we have a contradiction. If $d|N_r$, $d|N_{r+2}$ and $d \nmid N_{r+1}$ (which is equivalent to $d = 2$), then the contribution $\mathcal{R}$ is $r/(2r+1)$ and this is not zero because $r \geq 1$. If $d|N_{r+1}$, $d|N_{r+2}$ and $d \nmid N_r$, then $\mathcal{R} = 1/(4r+2)$. If $d|N_{r+2}$, $d \nmid N_r$ and $d \nmid N_{r+1}$, then $\mathcal{R} = -1/(4r+2)$. We conclude that $-1/2 - 1/(2r+1)$ is a pole of $Z_f^{(d)}(s)$ if and only if $d|4r+2$.
If $k = 2r$, $r \geq 2$, only the candidate pole associated to E_r can be a pole. If $d \nmid N_r$, then $\mathcal{R} = 0$. If $d|N_r$ and $d|N_{r-1}$, then $\mathcal{R} = (r-1)/(2r) \neq 0$. If $d|N_r$ and $d \nmid N_{r-1}$, then $\mathcal{R} = -1/(2r)$. Consequently, $-1/2 - 1/(2r)$ is a pole of $Z_f^{(d)}(s)$ if and only if $d|2r$.
Remark that we have proved that

$$\begin{aligned}\{s_0 \mid \exists f \in \mathcal{O}_2 \quad &: \quad \mathrm{mult}(f) = 2 \text{ and } Z_f^{(d)}(s) \text{ has a pole in } s_0\} \setminus \{-1/2\} \\ &= \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>2} \text{ and } d \mid \mathrm{lcm}(2,i) \right\}.\end{aligned}$$

(b) Suppose that $\mathrm{mult}(f) = 3$.

(b.1) Case $f_3 = xy(x+y)$. We get that $-1/2 - 1/6$ is a pole if and only if $d|3$ (which is equivalent to $d = 3$). This is consistent with the claim because $3 \mid \mathrm{lcm}(2,6)$.

(b.2) Case $f_3 = y^2(x+y)$. If $g(x) = 0$, then $Z_f^{(d)}(s)$ is identically zero (for every $d \geq 2$).
If $k = 2r+1$, only the candidate pole associated to E_r can be a pole. If $d|N_r$, then $d \nmid N_{r-1}$ because $d > 1$ and because N_r and N_{r-1} are odd numbers with difference 2. Consequently, if $d|2r+1$, then $\mathcal{R} = -1/(2r+1) \neq 0$.
If $k = 2r$, the only candidate pole which can be a pole is $-\nu_{r+1}/N_{r+1}$. If $d|N_{r+1}$ and $d|N_{r-1}$, then $d = 1$, which is a contradiction. So we have to consider two cases. If $d|N_{r+1}$, $d \nmid N_r$ and $d \nmid N_{r-1}$, then $\mathcal{R} = -1/(4r)$. If $d|N_{r+1}$, $d|N_r$ and $d \nmid N_{r-1}$, then $\mathcal{R} = 1/(4r)$. We obtain that $-1/2 - 1/(4r)$ is a pole if and only if $d|4r$.

(b.3) Case $f_3 = y^3$. We get analogously that if we have a pole, it is of the desired form. Remark that we only have to consider the case $r = 1$ in (b.3.i), (b.3.ii) and (b.3.iii).

(c) Suppose that $\mathrm{mult}(f) \geq 4$. As before we get that $Z_f^{(d)}(s)$ has no pole less than $-1/2$. $\square$

3 Surfaces

In this section, we prove the following theorem.

(3.0) Theorem *We have*

$$\mathcal{P}_3 \cap]-\infty, -1[= \left\{ -1 - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}.$$

Moreover, if $f \in \mathcal{O}_3$ has multiplicity 3 or more, then $Z_f(s)$ has no pole less than -1.

Remark. (i) It is a priori not obvious that the smallest value of $\mathcal{P}_3$ is $-3/2$. This is in contrast with the fact that it easily follows from lemma 2.5 that the smallest value of $\mathcal{P}_2$ is -1.

(ii) In (3.3.9) we give functions $f_k \in \mathcal{O}_3$ of arbitrary multiplicity such that $Z_{f_k}(s)$ has a pole in s_k, where $(s_k)_k$ is a sequence of real numbers larger than -1 and converging to -1. In particular $\mathcal{P}_3 \cap]-1, -41/42[\neq \emptyset$, which is in contrast to $\mathcal{T}_3 \cap]41/42, 1[= \emptyset$.

3.1 On candidate poles which are not poles

(3.1.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^3$ which satisfies $f(0) = 0$ and which is not identically zero. Let Y be the zero set of f. Fix an embedded resolution $g : X_t \to X_0 \subset \mathbb{C}^3$ for Y which is an isomorphism outside the singular locus of Y and which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : X_i \to X_{i-1}$ with irreducible nonsingular centre D_{i-1} and exceptional variety $E_i^{(0)}$ satisfying for $i = 0, \ldots, t-1$:

(a) the codimension of D_i in X_i is at least 2;

(b) D_i is a subset of the strict transform of Y under $g_1 \circ \cdots \circ g_i$;

(c) the union of the exceptional varieties of $g_1 \circ \cdots \circ g_i$ has only normal crossings with D_i, i.e., for all $P \in D_i$, there are three surface germs through P which are in normal crossings such that each exceptional surface germ through P is one of them and such that the germ of D_i at P is the intersection of some of them;

(d) the origin 0 of $\mathbb{C}^3$ is an element of $(g_1 \circ \cdots \circ g_i) D_i$; and

(e) D_i contains a point in which $(g_1 \circ \cdots \circ g_i)^{-1} Y$ has no normal crossings.

Remark that such a resolution always exists by Hironaka's theorem [Hi].

(3.1.2) Fix an exceptional variety $E_i^{(0)}$. The strict transform E_i of $E_i^{(0)}$ in X_t is obtained by a finite succession of blowing-ups h_j, $j \in T_e := \{1, \ldots, m\}$,

$$E_i^{(0)} \xleftarrow{h_1} E_i^{(1)} \xleftarrow{h_2} \cdots E_i^{(j-1)} \xleftarrow{h_j} E_i^{(j)} \cdots \xleftarrow{h_{m-1}} E_i^{(m-1)} \xleftarrow{h_m} E_i^{(m)} = E_i$$

with centre $P_{j-1} \in E_i^{(j-1)}$ and exceptional curve $C_j^{(j)}$. The irreducible components of the intersection of $E_i^{(0)}$ with irreducible components of $(g_1 \circ \cdots \circ g_i)^{-1} Y$ different from $E_i^{(0)}$ are denoted by $C_j^{(0)}$, $j \in T_s$. The strict transform of $C_j^{(k)}$ in $E_i^{(l)}$ is denoted (whenever this makes sense) by $C_j^{(l)}$ and we set $C_j = C_j^{(m)}$. Remark that $h := h_1 \circ \cdots \circ h_m$ is an embedded resolution of $\cup_{j \in T_s} C_j^{(0)}$. For each $j \in T := T_s \cup T_e$ the curve C_j is an irreducible component of the intersection of E_i with exactly one other component of $g^{-1}Y$. Let this component have numerical data (N_k, ν_k) and set $\alpha_j = \nu_k - (\nu_i/N_i)N_k$.

(3.1.3) Suppose that $E_i^{(0)} \subset (g_1 \circ \cdots \circ g_i)^{-1}\{0\}$ and that $\alpha_j \neq 0$ for every $j \in T$. The contribution $\mathcal{R}$ of E_i to the residue of $Z_f(s)$ at the candidate pole $-\nu_i/N_i$ is

$$\frac{1}{N_i} \left(\sum_{I \subset T} \chi(\overset{\circ}{C}_I) \prod_{j \in I} \alpha_j^{-1} \right), \tag{7}$$

where $\overset{\circ}{C}_I$ denotes the subset $(\cap_{j \in I} C_j) \setminus (\cup_{j \notin I} C_j)$ of E_i. Remark that $\overset{\circ}{C}_\emptyset = E_i \setminus (\cup_{j \in T} C_i)$. We now state some relations between the α_i, which will allow us to prove that this contribution is identically zero (i.e., zero for any value of the alphas) for a lot of intersection configurations on $E_i^{(0)}$.

(3.1.4) To the creation of $E_i^{(0)} \subset (g_1 \circ \cdots \circ g_i)^{-1}\{0\}$ in the resolution process, we associate the relation

$$\sum_{j \in T_s} d_j(\alpha_j - 1) + 3 - \dim D_{i-1} = 0, \tag{8}$$

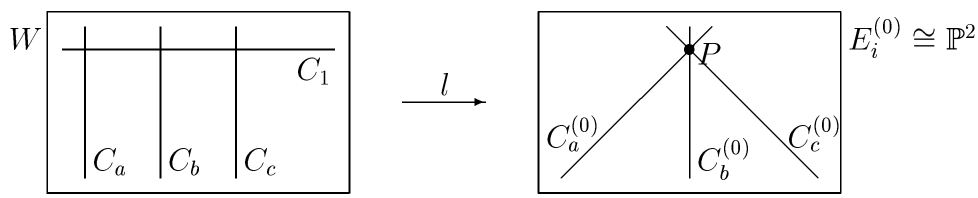
where d_i, $i \in T_s$, is the degree of the intersection cycle $C_i^{(0)} \cdot F$ on F for a general fibre F of $g_i|_{E_i^{(0)}} : E_i^{(0)} \to D_{i-1}$ over a point of D_{i-1}. In particular, when D_{i-1} is a point, we have that $E_i^{(0)} \cong \mathbb{P}^2$ and that d_i is just the degree of the curve $C_i^{(0)}$. To the blowing-up h_j we associate the relation

$$\alpha_j = \sum_{k \in T_s \cup \{1, \ldots, j-1\}} \mu_k(\alpha_k - 1) + 2, \tag{9}$$

where μ_k, $k \in T_s \cup \{1, \ldots, j-1\}$, is the multiplicity of P_{j-1} on $C_k^{(j-1)}$. See [Ve1] for more general statements in arbitrary dimension and proofs.

(3.1.5) Now we proceed in the same way as in [Ve3] for Igusa's p-adic zeta function. One easily verifies that the number (7) does not change when we do an extra blowing-up h_{m+1} at a point $P_m \in E_i^{(m)}$ and associate to the new exceptional curve a number α using (9). Because of this observation, one can compute $\mathcal{R}$ if one has the curves $C_j^{(0)}$, $j \in T_s$, on $E_i^{(0)}$ together with the associated values α_j as follows. Compute the *minimal* embedded resolution of $\cup_{j\in T_s} C_j^{(0)}$ and compute the alpha associated to an exceptional curve using (9). By putting these data in (7), we get $\mathcal{R}$.

(3.1.6) *Example.* Suppose that $E_i^{(0)}$ is the exceptional variety of a blowing-up at a point and suppose that the intersection configuration on $E_i^{(0)}$ consists of three projective lines $C_j^{(0)}$, $j \in T_s := \{a, b, c\}$, all passing through the same point P. Suppose that $\alpha_j \neq 0$ for all $j \in T$. The minimal embedded resolution $l : W \to E_i^{(0)}$ is the blowing-up at P. By abuse of notation, we denote the exceptional curve by C_1 and the strict transform of $C_j^{(0)}$, $j \in T_s$, by C_j.



By relations (8) and (9) we have $\alpha_a + \alpha_b + \alpha_c = 0$ and $\alpha_1 = \alpha_a + \alpha_b + \alpha_c - 1 = -1$ respectively. Now we can calculate the contribution $\mathcal{R}$ of the strict transform of $E_i^{(0)}$ in X_t to the residue of $Z_f(s)$ at the candidate pole $-\nu_i/N_i$:

$$\begin{aligned} \mathcal{R} &= \frac{1}{N_i}\left(\sum_{I\subset T} \chi(\overset{\circ}{C}_I) \prod_{j\in I} \alpha_j^{-1}\right) \\ &= \frac{1}{N_i}\left(-1 - \frac{1}{\alpha_1} + \frac{1}{\alpha_a} + \frac{1}{\alpha_b} + \frac{1}{\alpha_c} + \frac{1}{\alpha_1\alpha_a} + \frac{1}{\alpha_1\alpha_b} + \frac{1}{\alpha_1\alpha_c}\right) \\ &= 0. \end{aligned}$$

We stress that $\mathcal{R}$ is zero for any possible value of α_a, α_b and α_c.

3.2 Multiplicity 2

(3.2.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^n$ which satisfies $f(0) = 0$, and let F be the germ of the holomorphic function $f + x_{n+1}^2$ on a neighbourhood of the origin 0 in $\mathbb{C}^{n+1}$. Then the following equality is obtained in [ACLM], see also the Thom-Sebastiani principle in [DL3]:

$$Z_F(s) = \frac{1}{2s+1} + \frac{s(2s+3)}{2(s+1)(2s+1)} Z_f\left(s + \frac{1}{2}\right) - \frac{3s}{2(s+1)} Z_f^{(2)}\left(s + \frac{1}{2}\right).$$

(3.2.2) Proposition. *The set*

$$\{s_0 \mid \exists f \in \mathcal{O}_3 : \operatorname{mult}(f) = 2 \textit{ and } Z_f(s) \textit{ has a pole in } s_0\} \cap \,]-\infty, -1[$$

is equal to

$$\left\{ -1 - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}.$$

Proof. Let f be an element of $\mathcal{O}_3$ with multiplicity 2. Up to an affine coordinate transformation, the part of degree two in the Taylor series of f is equal to x^2, $x^2 + y^2$ or $x^2 + y^2 + z^2$. Using (2.7), we may suppose that f is of the form $x^2 + g(y, z)$ with $g(y, z) \in \mathcal{O}_2$. The formula in (3.2.1) and the result for curves imply that every pole of $Z_f(s)$ less than -1 is of the form $-1 - 1/i$, $i \in \mathbb{Z}_{>1}$. For the other inclusion, we remark that the poles of the local topological zeta function associated to $x^2 + y^2 + z^i$, $i \geq 2$, are $-1 - 1/i$ and -1. $\square$

(3.2.3) Our next goal is to give a sequence of poles larger than -1 and converging to -1. Keeping in mind the formula in (3.2.1), we try to find functions $f_k \in \mathcal{O}_2$ such that $Z_{f_k}(s)$ has a pole in s_k, where $(s_k)_k$ is a sequence of real numbers larger than $-1/2$ and converging to $-1/2$. Set $f_k = x^3y^2 + x^k$ for $k \geq 5$.
We obtain the following equalities after some calculations:

$$Z_{f_{2r+4}}(s) = \frac{3s^2 + 2rs + 8s + 2r + 3}{(4rs + 8s + 2r + 3)(3s + 1)(s + 1)}\,, \quad Z^{(2)}_{f_{2r+4}}(s) = \frac{1}{4rs + 8s + 2r + 3}\,,$$

$$Z_{f_{2r+3}}(s) = \frac{3s^2 - rs - 2s - r - 1}{(2rs + 3s + r + 1)(3s + 1)(s + 1)}\,, \quad Z^{(2)}_{f_{2r+3}}(s) = 0.$$

Now we use the formula in (3.2.1) to calculate the local topological zeta function of $F_k := f_k + z^2$. We obtain for even and odd k that

$$Z_{F_k}(s) = \frac{(6k - 6)s^2 + (15k - 5)s + 10k - 5}{(6s + 5)(s + 1)(2ks + 2k - 1)}.$$

Finally, we make the substitution $s = -(2k - 1)/(2k)$ in the numerator in order to check that this value, which converges to -1 if k goes to infinity, is a pole. We obtain

$$\frac{(k - 1)(k - 3)(2k - 1)}{2k^2}.$$

This value never becomes zero because $k \geq 5$. Consequently, $-(2k - 1)/(2k)$ is always a pole of $Z_{F_k}(s)$.

Remark. In particular we obtain that $\mathcal{P}_3 \cap]-1, -41/42[\neq \emptyset$, which is in contrast to $\mathcal{T}_3 \cap]41/42, 1[= \emptyset$.

3.3 Multiplicity larger than 2

(3.3.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^3$ which satisfies $f(0) = 0$ and which is not identically zero. Let Y be the zero set of f. Fix an embedded resolution g for Y which is a composition of blowing-ups $g_{ij} : X_i \to X_j$ with irreducible nonsingular centre D_j and exceptional surface E_i as in (3.1.1). Denote the irreducible components of Y by E_i, $i \in T_s$. The strict transform of a variety E_i by a succession of blowing-ups will be denoted in the same way. The numerical data of E_i are (N_i, ν_i).

(3.3.2) The following table gives the numerical data of E_i. In the columns, the dimension of D_j is kept fixed. In the rows, the number of exceptional surfaces through D_j is kept fixed. So E_k, E_l and E_m represent exceptional surfaces that contain D_j. The multiplicity of D_j on the strict transform of Y is denoted by μ_{D_j}.

	D_j is a point P	D_j is a curve L
/	$(\mu_P, 3)$	$(\mu_L, 2)$
E_k	$(N_k + \mu_P, \nu_k + 2)$	$(N_k + \mu_L, \nu_k + 1)$
E_k and E_l	$(N_k + N_l + \mu_P, \nu_k + \nu_l + 1)$	$(N_k + N_l + \mu_L, \nu_k + \nu_l)$
E_k, E_l and E_m	$(N_k + N_l + N_m + \mu_P, \nu_k + \nu_l + \nu_m)$	/

(3.3.3) Lemma. *Suppose that* $\mathrm{mult}(f) \geq 3$. *If there is no exceptional surface through* D_j, *then* $-\nu_i/N_i \geq -1$.

Proof. The case that the centre D_j is a point P through which no exceptional surface passes can only occur in the first blowing-up because of condition (d) in (3.1.1) and because the inverse image of 0 in X_j is contained in the union of the exceptional surfaces in X_j. Since $\mathrm{mult}(f) \geq 3$, we have in this case $-\nu_i/N_i = -3/\mu_P = -3/\mathrm{mult}(f) \geq -1$.
If the centre D_j is a curve L contained in no exceptional surface, then $\mu_L \geq 2$ because our embedded resolution is an isomorphism outside the singular locus of Y. Consequently, we get in this case $-\nu_i/N_i = -2/\mu_L \geq -1$. $\square$

(3.3.4) Suppose that D_j is contained in at least one exceptional surface and that the candidate poles associated to the exceptional surfaces that pass through D_j are larger than or equal to -1. Then the table in (3.3.2) implies that also $-\nu_i/N_i \geq -1$, unless D_j is a nonsingular point P of the strict transform of Y through which only one exceptional surface E_0 passes and $-\nu_0/N_0 = -1$. Suppose that we are in this situation. Denote the unique irreducible component of the strict transform of Y which passes through P by E_a. Consider now a small enough neighbourhood Z_0 of P on which E_a is nonsingular such that, if we restrict the blowing-ups g_{ij} to the inverse image of Z_0, we get an embedded resolution $h = h_1 \circ \cdots \circ h_s$ for the germ of $E_a \cup E_0$ at P which is a composition of blowing-ups $h_i : Z_i \to Z_{i-1}$, $i \in \{1, \ldots, s\}$, with irreducible nonsingular centre $D'_{i-1} := D_{i-1} \cap Z_{i-1}$ and exceptional surface $E'_i := E_i \cap Z_i$ satisfying for $i = 0, \ldots, s-1$:

(a) the codimension of D'_i in Z_i is at least 2;
(b) D'_i is a subset of $E'_a := E_a \cap Z_i$;

(c) $\cup_{l\in\{0,1,\ldots,i\}} E'_l$ has only normal crossings with D'_i, where $E'_0 := E_0 \cap Z_0$;
(d) the image of D'_i under $h_1 \circ \cdots \circ h_i$ contains P; and
(e) if $D_i = D'_i$, then D_i contains a point where there are no normal crossings.

Remark that it can happen that g_{ij} is an isomorphism on the inverse image of Z_0. Because we did not specify the indices in (3.3.1), we were able to get a nice notation here. Remark also that $D_i = D'_i$ if D_i is a point. From now on, we study the resolution $h : Z_s \to Z_0$ for the germ of $E_a \cup E_0$ at P.

(3.3.5) Lemma. *If $D_i = D'_i$, then D_i is a subset of E'_0.*

Proof. Remark that D_i has to lie in an exceptional surface because E'_a is nonsingular and because an embedded resolution is an isomorphism outside the singular locus of Y.

First we consider the case that $D_i = D'_i$ is a point contained in exceptional surfaces different from E'_0 and in the surface E'_a. The union of these surfaces has normal crossings at D_i because E'_a, considered as a subset of Z_0, is nonsingular. This is in contradiction with (e). Remark that it can thus not happen that E'_a and three exceptional surfaces different from E'_0 have a point in common.

The case that $D_i = D'_i$ is a curve contained in exactly two exceptional surfaces different from E'_0 and in the surface E'_a cannot occur because E'_a is a nonsingular subset of Z_0 and therefore these three surfaces should have normal crossings.

Finally we study the case that $D_i = D'_i$ is a curve contained in one exceptional surface E'_j different from E'_0 and in E'_a. Condition (c) implies that every point of D_i is contained in at most one exceptional surface different from E'_j. Moreover, such an exceptional surface has to be transversal to D_i. This implies that there are normal crossings at every point of D_i, which is in contradiction with (e). Therefore, this case cannot occur. $\square$

(3.3.6) Lemma. *Suppose that* $\mathrm{mult}(f) \geq 3$. *Then we have $\nu_i \leq N_i + 1$ for every exceptional surface E_i, $i \in \{1, \ldots, s\}$. Moreover, $\nu_i = N_i + 1$ if and only if D_{i-1} is a point and the numerical data of every exceptional surface E_j different from E_0 and through D_{i-1} satisfy $\nu_j = N_j + 1$.*

Proof. The proof is by induction on i. Since $\nu_0 = N_0$, we have that $\nu_1 = N_1 + 1$. Suppose now that $\nu_j \leq N_j + 1$ for every exceptional surface E_j through D_{i-1}.

Case 1: D_{i-1} is a point. We obtain from (3.3.5) that D_{i-1} is a subset of E'_0. Because $\nu_0 = N_0$ and because every other exceptional surface E_j through D_{i-1} satisfies $\nu_j \leq N_j + 1$, the table of (3.3.2) gives us that $\nu_i \leq N_i + 1$.

Case 2: D_{i-1} is a curve. If $D_{i-1} \neq D'_{i-1}$, then $D'_{i-1} \not\subset (h_1 \circ \cdots \circ h_{i-1})^{-1}P$ and therefore we get as in the beginning of (3.3.4) that $-\nu_i/N_i \geq -1$. If $D_{i-1} = D'_{i-1}$, one computes from (3.3.2) and the previous lemma that $-\nu_i/N_i \geq -1$.

We have now proved the first part of the lemma. Using this first part and the table of (3.3.2), we get the second part. $\square$

(3.3.7) Lemma. *If* $\operatorname{mult}(f) \geq 3$ *and if the numerical data of* E_i *satisfy* $\nu_i = N_i + 1$, *then* $-\nu_i/N_i \neq -\nu_j/N_j$ *for every exceptional surface* E_j *that intersects* E_i *at some stage of the resolution process.*

Proof. Let E_j be an exceptional surface that intersects E_i at some stage of the resolution process. If E_j is created before E_i, then E_j contains the point D_{i-1}. Otherwise, E_j is created by a blowing-up at a point of E_i or by a blowing-up along a curve.

If E_j is created by a blowing-up along a curve, then $-\nu_j/N_j \geq -1$, and consequently $-\nu_i/N_i \neq -\nu_j/N_j$. Now we consider the case that E_j contains the point D_{i-1}. There is no problem if $\nu_j \leq N_j$. Consequently, suppose that $\nu_j = N_j + 1$. From the table in (3.3.2), we get $N_j < N_i$. Therefore, $-\nu_i/N_i = -(N_i+1)/N_i > -(N_j+1)/N_j = -\nu_j/N_j$. The case that E_j is created by a blowing-up at a point of E_i is treated analogously. $\square$

(3.3.8) Proposition. *If* $\operatorname{mult}(f) \geq 3$, *then no pole of* $Z_f(s)$ *is less than* -1.

Proof. Suppose that $\operatorname{mult}(f) \geq 3$.

We have only to consider exceptional surfaces with a candidate pole less than -1. Recall from (3.3.6) that $-\nu_i/N_i < -1$ if and only if D_{i-1} is a point and all exceptional surfaces through the point D_{i-1} different from E_0 have a candidate pole less than -1. We will determine all possible intersection configurations on such surfaces just after their creation.

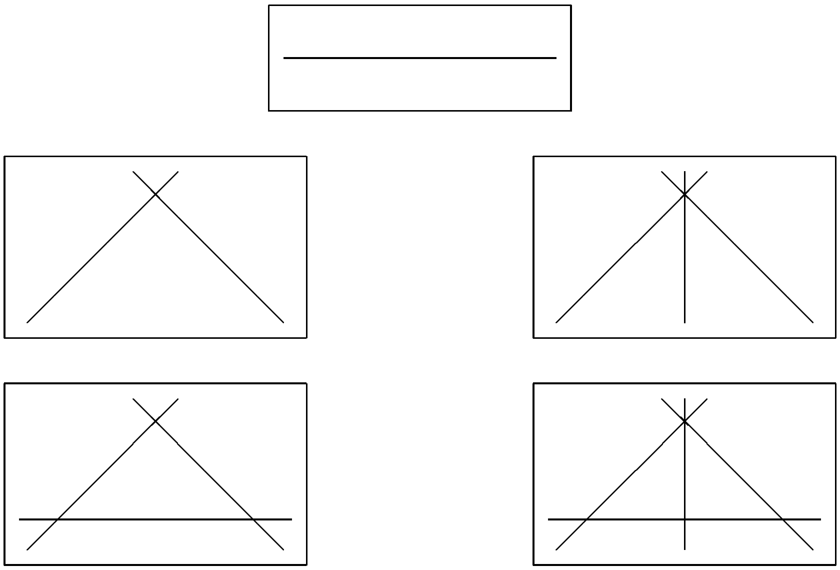
If $-\nu_i/N_i \geq -1$ and $-\nu_{i+1}/N_{i+1} < -1$, then the blowing-ups along D_{i-1} and D_i commute with each other. Therefore, we may assume that there is a k (larger than zero because $-\nu_1/N_1 < -1$) such that $-\nu_i/N_i < -1$ for $1 \leq i \leq k$ and $-\nu_i/N_i \geq -1$ for $k < i \leq s$.

The intersection configuration on E_1 consists of one projective line, which is the intersection with E_0 and E_a. The points of Z_1 in which we do not have normal crossings and which lie above P are those on this projective line. This implies the following statement for $i = 2$.

> If Q is a point of Z_{i-1}, $i \in \{2, \ldots, k\}$, in which we do not have normal crossings and which lies above P (so consequently Q is a point of E_0, of one or two other exceptional surfaces and of E_a), then there exists an exceptional surface E_l through Q with the property $E_0 \cap E_l = E_a \cap E_l$. $(*)$

We prove this statement by induction on i. Suppose that it is true for $i = j \in \{2, \ldots, k-1\}$. We give the proof for $i = j+1$. The statement follows from the induction hypothesis for points not on E_j, because a blowing-up is an isomorphism outside the exceptional surface. So we prove it for points on E_j. By the induction hypothesis applied to the point D_{j-1}, we obtain that there exists an exceptional surface E_l through D_{j-1} such that $E_0 \cap E_l = E_a \cap E_l$ in Z_{j-1}. But then $E_a \cap E_l = E_0 \cap E_l$ in Z_j, which solves the problem for the point $E_0 \cap E_l \cap E_j$. There are other points on E_j in which we do not have normal crossings if and only if E_a is tangent to E_0 in D_{j-1}. In this case, the points in which we do not have normal crossings are the points of $E_0 \cap E_j$. Because $E_0 \cap E_j = E_a \cap E_j$, we are done.

Because the centre of a blowing-up satisfies the conditions of the statement, we obtain that the possible intersection configurations are the following configurations of lines in $\mathbb{P}^2$:



For all these configurations, we can calculate as in (3.1.6) that the contribution to the residue is 0. The second author did this already in [Ve3] for Igusa's p-adic zeta function. The point is that $(*)$ excludes the configuration consisting of four lines in general position, for which this contribution is not zero. Remark also that we need here that the alphas are not zero, a fact we proved in (3.3.7). $\square$

(3.3.9) In (3.2.3), we found functions $f_k \in \mathcal{O}_3$ of multiplicity 2 such that $Z_{f_k}(s)$ has a pole in s_k, where $(s_k)_k$ is a sequence of real numbers larger than -1 and converging to -1. Here we construct for *every* $n \geq 0$ functions $f_k \in \mathcal{O}_3$ of multiplicity $n+2$ with this property. We use the formula obtained by Denef and Loeser in [DL1, Théorème 5.3], which expresses the local topological zeta function of a non-degenerated polynomial in terms of its Newton polyhedron. Fix $n \geq 0$ and set $f_k = x^n z^2 + x^{3+n} y^2 + x^k$ for $k \geq n+4$. Then

$$Z_{f_k}(s) = \frac{\begin{array}{c}(-2n^2-6n)s^3 + (n^2+3kn-4n+6k-6)s^2 \\ +(-4n^2+4kn-7n+15k-5)s - 10n + 10k - 5\end{array}}{(6s+2ns+5)(s+1)(2ks+2k-2n-1)(ns+1)}.$$

Consequently, $-(2k-2n-1)/(2k)$ is a pole if and only if it is not a zero of the numerator. So we make the substitution $s = -(2k-2n-1)/(2k)$ in the numerator and obtain

$$\frac{(k-1-2n)(k-n-3)(2k-2n-1)(2n^2-2kn+n+2k)}{4k^3}.$$

Because $k \geq n+4$, this is zero if and only if $k = 1+2n$. Thus we have found for any multiplicity larger than one a sequence with the desired property.

4 Other zeta functions

(4.1) Denef and Loeser associate in [DL2] to a polynomial its motivic zeta function, which is a much finer invariant than its topological zeta function. Instead of the usual topological Euler-Poincaré characteristic, it involves the so-called universal Euler characteristic of an algebraic variety, i.e., its class in the Grothendieck ring.

We recall this notion. The Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex algebraic varieties is the free abelian group generated by the symbols $[V]$, where V is a variety, subject to the relations $[V] = [V']$, if V is isomorphic to V', and $[V] = [V \setminus W] + [W]$, if W is closed in V. Its ring structure is given by $[V] \cdot [W] := [V \times W]$. We set $\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}]$ and denote by $\mathcal{M}$ the localization of $K_0(\mathrm{Var}_{\mathbb{C}})$ with respect to $\mathbb{L}$.

(4.2) In [DL2] the motivic zeta function is more generally defined for a regular function f on a smooth algebraic variety X, with respect to a subvariety W of X; we refer to [DL2, section 2] for this definition. One easily verifies that the construction is still valid for a germ f of a holomorphic function at $0 \in \mathbb{C}^n$ when $W = \{0\}$; we denote this (local) motivic zeta function by $Z_{\mathrm{mot},0,f}(s)$. Then, with the notation of (1.1), the formula of [DL2, Theorem 2.2.1] yields that

$$Z_{\mathrm{mot},0,f}(s) = \mathbb{L}^{-n} \sum_{I \subset T} [\overset{\circ}{E}_I \cap g^{-1}\{0\}] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i + sN_i} - 1}.$$

Here $\mathbb{L}^{-s}$ should be considered as a variable, and this expression lives in a localization of the polynomial ring $\mathcal{M}[\mathbb{L}^{-s}]$.

(4.3) The motivic zeta function $Z_{\mathrm{mot},0,f}(s)$ specializes to $Z_{\mathrm{top},0,f}(s)$ [DL2, subsection 2.3], but also to various 'intermediate level' zeta functions. An important one uses Hodge polynomials. Recall that the Hodge polynomial of a complex algebraic variety V is

$$H(V) = H(V,u,v) := \sum_{p,q} \left(\sum_{i \geq 0} (-1)^i h^{p,q}\left(H^i_c(V,\mathbb{C})\right) \right) u^p v^q \in \mathbb{Z}[u,v],$$

where $h^{p,q}\left(H_c^i(V,\mathbb{C})\right)$ is the rank of the (p,q)-Hodge component of the i-th cohomology group with compact support of V. The zeta function of f on this level is

$$Z_{\mathrm{Hod},0,f}(s) = (uv)^{-n} \sum_{I\subset T} H\left(\overset{\circ}{E}_I \cap g^{-1}\{0\}\right) \prod_{i\in I} \frac{uv-1}{(uv)^{\nu_i+sN_i}-1};$$

here $(uv)^{-s}$ is a variable, and this zeta function lives e.g. in the field of rational functions in $(uv)^{-s}$ over $\mathbb{Q}(u,v)$.

(4.4) As in [RV] we define the poles of $Z_{\mathrm{Hod},0,f}(s)$ to be the real numbers s_0 such that $(uv)^{-s_0}$ is a pole of $Z_{\mathrm{Hod},0,f}(s)$, considered as rational function in $(uv)^{-s}$. Then we have the following.

Theorems 2.9 and 3.0 are still valid with $Z_f(s) = Z_{\mathrm{top},0,f}(s)$ replaced by $Z_{\mathrm{Hod},0,f}(s)$ and $\mathcal{P}_n = \{s_0 \mid \exists f \in \mathcal{O}_n \ : \ Z_{\mathrm{Hod},0,f}(s) \text{ has a pole in } s_0\}$. The proofs are the same as before; they essentially just use the 'geometry' of a resolution.

A good definition of poles of $Z_{\mathrm{mot},0,f}(s)$ is not immediately clear, due to the fact that $\mathcal{M}$ could have zero divisors (at present this is an open question). Using the definition of [RV] for real poles, Theorems 2.9 and 3.0 are also valid for $Z_{\mathrm{mot},0,f}(s)$.

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