

# ON THE POLES OF MAXIMAL ORDER OF THE TOPOLOGICAL ZETA FUNCTION

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**ABSTRACT.** The global and local topological zeta functions are singularity invariants associated to a polynomial  $f$  and its germ at 0, respectively. By definition these zeta functions are rational functions in one variable and their poles are negative rational numbers. In this paper we study their poles of maximal possible order. When  $f$  is non degenerate with respect to its Newton polyhedron we prove that its local topological zeta function has at most one such pole, in which case it is also the largest pole; concerning the global zeta function we give a similar result. Moreover for any  $f$  we show that poles of maximal possible order are always of the form  $-1/N$  with  $N$  a positive integer.

## INTRODUCTION

(0.1) To  $f \in \mathbb{C}[x_1, \dots, x_n]$  is associated a singularity invariant, called the *topological zeta function* of  $f$ , which is expressed as follows in terms of an embedded resolution of  $f^{-1}\{0\} \subset \mathbb{A}^n$ . For simplicity of notation suppose that  $f(0) = 0$ .

Let  $h : X \rightarrow \mathbb{A}^n$  be an embedded resolution of  $f^{-1}\{0\}$ . We denote by  $E_i, i \in S$ , the irreducible components of  $h^{-1}(f^{-1}\{0\})$ , and by  $N_i$  and  $\nu_i - 1$  the multiplicities of  $E_i$  in the divisor on  $X$  of  $f \circ h$  and  $h^*(dx_1 \wedge \dots \wedge dx_n)$ , respectively. The  $(N_i, \nu_i), i \in S$ , are called the *numerical data* of the resolution  $(X, h)$ . For  $I \subset T$  we denote also  $E_I := \cap_{i \in I} E_i$  and  $E_I^\circ := E_I \setminus (\cap_{j \notin I} E_j)$ .

**Definition.** Let  $\chi(\cdot)$  denote the topological Euler–Poincaré characteristic. To  $f$  and  $d \in \mathbb{N} \setminus \{0\}$  one associates the rational functions in one variable

$$Z_{\text{top}}(s) = Z_{\text{top}}^{(d)}(s) := \sum_{\substack{I \subset S \\ \forall i \in I: d | N_i}} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{N_i s + \nu_i}$$

and

$$Z_{\text{top},0}(s) = Z_{\text{top},0}^{(d)}(s) := \sum_{\substack{I \subset S \\ \forall i \in I: d | N_i}} \chi(E_I^\circ \cap h^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i}$$

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1991 *Mathematics Subject Classification.* 14B05 14E15 32S50 (32S45).

*Key words and phrases.* Singularity invariant, topological zeta function.

The second author is a Postdoctoral Fellow of the Belgian National Fund for Scientific Research (F.W.O.)

which are both called the topological zeta function of  $f$ , more precisely the global and local one, respectively. They are invariants of  $f$  and the germ of  $f$  at 0, respectively, and were introduced by Denef and Loeser in [DL1]. The remarkable fact that these expressions do not depend on the chosen resolution was originally proved by writing them as a certain limit of Igusa's local zeta functions [DL1]; it also follows by considering the topological zeta function as a specialization of the recently introduced motivic Igusa zeta functions, see [DL2, (2.3)].

(0.2) In particular the poles of the topological zeta function of  $f$  are interesting invariants, and various conjectures relate them to the eigenvalues of local monodromy of  $f$ , see for example [DL1, Ve3]. In this paper we study the poles of maximal possible order, i.e. of order  $n$ . Concerning the local topological zeta function there is the following conjecture of the second author, which he proved for  $n = 2$  in [Ve2, Theorem 4.2].

**Conjecture.** (i)  $Z_{\text{top},0}(s)$  has at most one pole of order  $n$ , and  
(ii) if  $Z_{\text{top},0}(s)$  has in  $\tilde{s}$  a pole of order  $n$ , then  $\tilde{s}$  is the largest pole of  $Z_{\text{top},0}(s)$ .

Remark that in any case the largest candidate pole of  $Z_{\text{top},0}^{(1)}(s)$  is just the so-called *log canonical threshold* of  $f$  at 0, denoted  $c_0(f)$ , see [K]. We have that  $-c_0(f)$  is the largest root of the Bernstein–Sato polynomial of  $f$ , and if 0 is an isolated singularity of  $f^{-1}\{0\}$ , then  $c_0(f) = \min\{1, \alpha_f\}$ , where  $\alpha_f$  is Arnold's complex singularity index [AVG].

(0.3) We will prove that the conjecture above is true for polynomials  $f$  which are non degenerate with respect to their Newton polyhedron at the origin. Remark that ‘almost all’ polynomials satisfy this property. And concerning the global topological zeta function we obtain the following.

**Proposition.** *Let  $f$  be non degenerate with respect to its global Newton polyhedron. Then*  
(i)  $Z_{\text{top}}(s)$  has at most 2 poles of order  $n$ , and  
(ii) if  $Z_{\text{top},0}(s)$  has in  $\tilde{s}$  a pole of order  $n$ , then  $\tilde{s} = -1$  or  $\tilde{s}$  is the largest pole of  $Z_{\text{top}}(s)$ .

Here we should remark that this last result is specific for non degenerate polynomials; it is not true for general  $f$ .

(0.4) Finally we show that any pole of order  $n$  of  $Z_{\text{top},0}(s)$  or  $Z_{\text{top}}(s)$  must be of the form  $-1/N$  with  $N \in \mathbb{N} \setminus \{0\}$ . This is an immediate corollary of the following geometrical result.

**Theorem.** *We use the notation of (0.1). Suppose that  $E_i, i \in I$ , are  $n$  different components of  $h^{-1}(f^{-1}\{0\})$ , such that  $\cap_{i \in I} E_i \neq \emptyset$  and  $\frac{\nu_i}{N_i} = t$  for all  $i \in I$ ; then  $t = \frac{1}{N}$  for some  $N \in \mathbb{N} \setminus \{0\}$ .*

(1.1) Let  $f \in \mathbb{C}[x_1, \dots, x_n]$  be a non-constant polynomial satisfying  $f(0) = 0$ . We write  $f = \sum_{k \in \mathbb{N}^n} a_k x^k$ , where  $k = (k_1, \dots, k_n)$  and  $x^k = x_1^{k_1} \cdots x_n^{k_n}$ ; then the support of  $f$  is  $\text{supp } f = \{k \in \mathbb{N}^n \mid a_k \neq 0\}$ . The *global Newton polyhedron*  $\Gamma_{gl}$  of  $f$  is the convex hull of  $\text{supp } f$ , and the *Newton polyhedron*  $\Gamma_0$  of  $f$  at the origin is the convex hull of  $\Gamma_{gl} + \mathbb{R}_+^n$ . ( $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ .)

**1.2. Definition.** One says that  $f$  is non degenerate with respect to  $\Gamma_{gl}$  and  $\Gamma_0$  if for every face  $\tau$  of  $\Gamma_{gl}$  (including  $\tau = \Gamma_{gl}$ ) and every *compact* face  $\tau$  of  $\Gamma_0$ , respectively, the polynomials  $f_\tau := \sum_{k \in \tau} a_k x^k$  and  $\partial f_\tau / \partial x_i, 1 \leq i \leq n$ , have no common zeroes in  $(\mathbb{C} \setminus \{0\})^n$ .

Roughly speaking, almost all polynomials are non degenerate with respect to  $\Gamma_{gl}$  or  $\Gamma_0$  [AVG, p.157].

(1.3) For  $a = (a_1, \dots, a_n) \in \mathbb{R}_+^n$  we put  $N(a) := \inf_{x \in \Gamma_0} a \cdot x$ ,  $\nu(a) := \sum_{i=1}^n a_i$  and  $F(a) := \{x \in \Gamma_0 \mid a \cdot x = N(a)\}$ . It is a fact that all  $F(a), a \neq 0$ , are faces of  $\Gamma_0$ . One associates to a face  $\tau$  of  $\Gamma_0$  a (dual) cone  $\tau^\circ \subset \mathbb{R}^n$ , defined as the closure in  $\mathbb{R}^n$  of  $\{a \in \mathbb{R}_+^n \mid F(a) = \tau\}$ . It is a rational convex cone, with vertex the origin, of dimension  $n - \dim \tau$ . In particular if  $\dim \tau = n - 1$  then  $\tau^\circ$  is a ray, say  $\tau^\circ = a\mathbb{R}^+$  for some  $a \in \mathbb{N}^n$ , and then the equation of the hyperplane through  $\tau$  is  $a \cdot x = N(a)$ . Also the map  $\tau \mapsto \tau^\circ$  is inclusion-reversing.

Finally we recall that faces of dimension  $n - 1$  of  $\Gamma_0$  are called *facets* and that every face  $\tau$  with  $\dim \tau < n$  is the intersection of the facets that contain  $\tau$ .

(1.4) We will recall below a formula for the topological zeta function when  $f$  is non degenerate. We first describe the terms that appear in this formula. Let  $C$  be a rational simplicial cone in  $\mathbb{R}_+^n$  (with vertex the origin); so it is of the form  $C = \mathbb{R}_+ a_1 + \cdots + \mathbb{R}_+ a_\ell$ , where  $a_1, \dots, a_\ell \in \mathbb{N}^n$  are linearly independent over  $\mathbb{R}$  and are primitive, i.e. with relatively prime components. We associate to  $C$  (and  $\Gamma_0$ ) the rational function

$$J_C(s) := \frac{\text{mult}(C)}{\prod_{i=1}^\ell (N(a_i)s + \nu(a_i))},$$

where  $\text{mult}(C) \in \mathbb{N} \setminus \{0\}$  is the *multiplicity* of  $C$ , whose definition is not important for our results, but is given for completeness in 1.8 below.

**1.5. Definition.** To an arbitrary face  $\tau$  of  $\Gamma_0$  we associate a rational function  $J_\tau(s)$  as follows.

- (i) If  $\tau = \Gamma_0$  we put  $J_\tau(s) := 1$ .
- (ii) Otherwise choose a decomposition  $\tau^\circ = \cup_{i=1}^r C_i$  of  $\tau^\circ$  in rational simplicial cones  $C_i$  of dimension  $\ell = \dim \tau^\circ$  such that  $\dim(C_i \cap C_j) < \ell$  if  $i \neq j$ . Then put  $J_\tau(s) := \sum_{i=1}^r J_{C_i}(s)$ .

As shown in [DL1, Lemme 5.1.1] the function  $J_\tau(s)$  above does not depend on the chosen decomposition of  $\tau^\circ$  and is thus well defined, and moreover the poles of  $J_\tau(s)$  are of the form  $-\nu(a)/N(a)$ , where  $a$  is orthogonal to a facet of  $\Gamma_0$  containing  $\tau$ .

**1.6. Theorem.** [DL1, Théorème 5.3] (i) If  $f$  is non degenerate with respect to  $\Gamma_0$ , then

$$Z_{\text{top},0}^{(1)}(s) = \sum_{\tau \text{ vertex of } \Gamma_0} J_\tau(s) + \left(\frac{s}{s+1}\right) \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_0, \\ \dim \tau \geq 1}} (-1)^{\dim \tau} (\dim \tau)! \text{Vol}(\tau) J_\tau(s)$$

and

$$Z_{\text{top},0}^{(d)}(s) = \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_0, \\ d|N(\tau^\circ)}} (-1)^{\dim \tau} (\dim \tau)! \text{Vol}(\tau) J_\tau(s) \quad \text{if } d > 1,$$

where  $N(\tau^\circ) := \text{lcd}_{a \in \tau^\circ \cap \mathbb{N}^n} N(a)$ , and where  $\text{Vol}(\tau) \in \mathbb{N} \setminus \{0\}$  is defined in 1.8 below.

(ii) If  $f$  is non degenerate with respect to  $\Gamma_{gl}$ , then there are analogous formulas for  $Z_{\text{top}}^{(1)}(s)$  and  $Z_{\text{top}}^{(d)}(s)$ ,  $d > 1$ , where the summation now runs over all faces of  $\Gamma_0$ .

Again we will not need the concrete meaning of  $\text{Vol}(\tau)$  for our results.

(1.7) Let  $t_0 = \min\{t \in \mathbb{R} | (t, \dots, t) \in \Gamma_0\}$  and let  $\tau_0$  denote the face of  $\Gamma_0$  that contains  $(t_0, \dots, t_0)$  in its relative interior. One can verify that  $s_0 := -\frac{1}{t_0}$  is the largest candidate-pole of  $Z_{\text{top},0}^{(d)}(s)$  or  $Z_{\text{top}}^{(d)}(s)$ , besides  $-1$  when  $d = 1$ .

(1.8) For the interested reader we recall here the definitions of the volume of a face and the multiplicity of a cone. Let  $\gamma$  be the convex hull in  $\mathbb{R}^n$  of a part of  $\mathbb{Z}^n$ . We denote by  $\omega_\gamma$  the volume form on  $\text{Aff}(\gamma)$ , the affine space spanned by  $\gamma$ , such that the parallelepiped spanned by a lattice-basis of  $\mathbb{Z}^n \cap \text{Aff}(\gamma)$  has volume 1.

(i) Let  $\tau$  be a face of  $\Gamma_0$ . If  $\dim \tau = 0$  we put  $\text{Vol}(\tau) := 1$ ; otherwise we define  $\text{Vol}(\tau)$  as the volume of  $\tau \cap \Gamma_{gl}$  for the volume form  $\omega_\tau$ . (When  $\tau$  is compact then  $\tau \cap \Gamma_{gl} = \tau$ .)

(ii) Let  $C$  be the  $\ell$ -dimensional rational simplicial cone in  $\mathbb{R}_+^n$  given by  $C = \mathbb{R}_+ a_1 + \dots + \mathbb{R}_+ a_\ell$ , where all  $a_i$  are primitive. Then  $\text{mult}(C)$  is the volume of the parallelepiped spanned by  $a_1, \dots, a_\ell$  for the volume form  $\omega_C$ .

## 2. DETERMINATION OF THE POLES OF MAXIMAL ORDER

Still using the notation of §1, we first derive some convex-geometric lemmas.

**Lemma 2.1.** Let  $V$  be a vertex of  $\Gamma_0$  such that  $J_V(s)$  has in  $\tilde{s}$  a pole of order  $n$ . Then  $V = (-\frac{1}{\tilde{s}}, \dots, -\frac{1}{\tilde{s}})$ .

*Proof.* We choose a decomposition of the dual cone  $V^\circ$  as described in Definition 1.5, and moreover without introducing new rays. This is always possible [DS, Lemme 2.3]. Take a simplicial cone  $C$  in this decomposition such that  $J_C(s)$  has in  $\tilde{s}$  a pole of order  $n$ , and let  $\xi_1, \dots, \xi_n$  be the primitive generators of  $C$  in  $\mathbb{N}^n$ . By the construction of the chosen decomposition we have that  $F(\xi_1), \dots, F(\xi_n)$  are facets of  $\Gamma_0$  containing  $V$ . Since  $J_C(s)$  has in  $\tilde{s}$  a pole of order  $n$ , we have moreover that  $\tilde{s} = -\frac{\nu(\xi_i)}{N(\xi_i)}$  for

$i = 1, \dots, n$ . Letting  $H_i$  denote the affine hyperplane through  $F(\xi_i)$ ,  $1 \leq i \leq n$  and  $\Delta = \{(t, \dots, t) | t \in \mathbb{R}\}$  the diagonal, we thus obtain that  $H_i \cap \Delta = (-\frac{1}{s}, \dots, -\frac{1}{s})$  for  $i = 1, \dots, n$ . From  $V = \cap_{i=1}^n F(\xi_i)$  we then derive

$$V \cap \Delta = \cap_{i=1}^n (F(\xi_i) \cap \Delta) \subset \cap_{i=1}^n (H_i \cap \Delta) = \{(-\frac{1}{s}, \dots, -\frac{1}{s})\},$$

and so  $V = (-\frac{1}{s}, \dots, -\frac{1}{s})$ .  $\square$

**Lemma 2.2.** *Let  $\gamma$  be a 1-dimensional face of  $\Gamma_0$  such that  $J_\gamma(s)$  has in  $-1$  a pole of order  $n - 1$ . Then  $(1, \dots, 1)$  is contained in the affine line  $L_\gamma$  through  $\gamma$ .*

*Proof.* As in the proof of Lemma 2.1 we choose a decomposition of  $\gamma^\circ$  as in Definition 1.5 without introducing new rays. Take a cone  $C$  in the decomposition such that  $J_C(s)$  has in  $-1$  a pole of order  $n - 1$ , and let  $\xi_1, \dots, \xi_{n-1}$  be the primitive generators of  $C$  in  $\mathbb{N}^n$ . Again  $F(\xi_1), \dots, F(\xi_{n-1})$  are facets of  $\Gamma_0$  containing  $\gamma$ , and here  $\nu(\xi_i) = N(\xi_i)$  for  $i = 1, \dots, n - 1$ . Letting  $H_i$  denote the affine hyperplane through  $F(\xi_i)$ , we thus have that  $(1, \dots, 1) \in H_i$  for  $i = 1, \dots, n - 1$ . Since  $\gamma = \cap_{i=1}^{n-1} F(\xi_i)$  and consequently  $L_\gamma = \cap_{i=1}^{n-1} H_i$  we obtain that  $(1, \dots, 1) \in L_\gamma$ .  $\square$

**Lemma 2.3.** *Let  $\gamma$  be a 1-dimensional compact face of  $\Gamma_0$  such that the affine line  $L_\gamma$  through  $\gamma$  contains  $(1, \dots, 1)$ . Then  $(1, \dots, 1) \in \gamma$ .*

*Proof.* Let  $V_1$  and  $V_2$  denote the vertices of  $\gamma$ . Let  $F_1, \dots, F_{n-1}$  be facets of  $\Gamma_0$  such that  $\gamma = \cap_{i=1}^{n-1} F_i$  and let  $\xi_i \in \mathbb{N}_+^n$  be such that  $F(\xi_i) = F_i$  for  $i = 1, \dots, n - 1$ . Since  $(1, \dots, 1) \in L_\gamma$  we have that  $\nu(\xi_i) = N(\xi_i)$  for  $i = 1, \dots, n - 1$ . Let now  $t \in \mathbb{R}$  be such that  $(1, \dots, 1) = tV_1 + (1 - t)V_2$ ; it suffices to prove that  $0 \leq t \leq 1$ .

We assume that  $V_1 \neq (1, \dots, 1)$  (the other case being trivial). We will denote the  $j$ th coordinate of  $a \in \mathbb{N}^n$  by  $(a)_j$ . Suppose that  $V_1 \in (1, \dots, 1) + \mathbb{R}_+^n$ ; then there exists some  $j$  in  $\{1, \dots, n\}$  such that  $(V_1)_j > 1$ . It then follows from  $V_1 \in \gamma$  and  $\nu(\xi_i) = N(\xi_i) = \xi_i \cdot V_1$  that  $(\xi_i)_j = 0$  for  $i = 1, \dots, n - 1$ . This contradicts the compactness of  $\gamma$ . Hence  $V_1 \notin (1, \dots, 1) + \mathbb{R}_+^n$ .

Let then  $j \in \{1, \dots, n\}$  be such that  $(V_1)_j = 0$ . Then  $1 = (1 - t)(V_2)_j$ , which immediately implies that  $0 < 1 - t = \frac{1}{(V_2)_j} \leq 1$ .  $\square$

*Remark.* As illustrated in Example 2.6, the condition compact in the statement of Lemma 2.3 cannot be omitted. Because of this fact, the proof we will give for Theorem 2.4 fails when we replace  $Z_{\text{top},0}(s)$  by  $Z_{\text{top}}(s)$ .

**2.4. Theorem.** *Let  $f$  be non degenerate with respect to  $\Gamma_0$ . Then*

- (i)  $Z_{\text{top},0}(s)$  has at most one pole of order  $n$ , and
- (ii) if  $Z_{\text{top},0}(s)$  has in  $\tilde{s}$  a pole of order  $n$ , then  $\tilde{s}$  is the largest pole of  $Z_{\text{top},0}(s)$ .

*Proof.* We will prove (ii) which immediately yields (i). So suppose that  $Z_{\text{top},0}^{(d)}(s)$  has in  $\tilde{s}$  a pole of order  $n$ ; then there is at least one term in the formula of Theorem 1.6 for  $Z_{\text{top},0}^{(d)}(s)$  that has in  $\tilde{s}$  a pole of order  $n$ .

Suppose first that there exists a vertex  $V$  of  $\Gamma_0$  such that  $J_V(s)$  has in  $\tilde{s}$  a pole of order  $n$  (and such that  $d|N(V^\circ)$ ). Then by Lemma 2.1 we must have  $V = (-\frac{1}{\tilde{s}}, \dots, -\frac{1}{\tilde{s}})$ ; so  $V = \tau_0$  and  $\tilde{s} = s_0$ .

Suppose on the other hand that there is no such vertex of  $\Gamma_0$ . Then necessarily  $d = 1$  and  $\tilde{s} = -1$  and there must exist a compact 1-dimensional face  $\gamma$  of  $\Gamma_0$  such that  $J_\gamma(s)$  has in  $-1$  a pole of order  $n - 1$ . Then Lemmas 2.2 and 2.3 imply that  $(1, \dots, 1) \in \gamma$ , and thus  $\tau_0 \subset \gamma$  and  $s_0 = -1 = \tilde{s}$ .  $\square$

So we proved Conjecture 0.2 for non degenerate polynomials. Analogous arguments yield the following result concerning the global zeta function  $Z_{\text{top}}(s)$ .

**2.5. Proposition.** *Let  $f$  be non degenerate with respect to  $\Gamma_{gl}$ . Then*

- (i)  $Z_{\text{top}}(s)$  has at most 2 poles of order  $n$ , and
- (ii) if  $Z_{\text{top},0}(s)$  has in  $\tilde{s}$  a pole of order  $n$ , then  $\tilde{s} = -1$  or  $\tilde{s}$  is the largest pole of  $Z_{\text{top}}(s)$ .

*Proof.* We only have to prove (ii). Suppose that  $Z_{\text{top}}^{(d)}(s)$  has in  $\tilde{s} \neq -1$  a pole of order  $n$ . Then there is a vertex  $V$  of  $\Gamma_0$  such that  $J_V(s)$  has in  $\tilde{s}$  a pole of order  $n$  (and such that  $d|N(V^\circ)$ ). By Lemma 2.1 we have that  $V = (-\frac{1}{\tilde{s}}, \dots, -\frac{1}{\tilde{s}})$  and so  $V = \tau_0$  and  $\tilde{s} = s_0$ .  $\square$

**2.6. Example.** Take  $f = x^2y^2 + x^4y + xy^4 + xy^5 = xy(xy + x^3 + y^3 + y^4)$ . Its Newton polyhedron  $\Gamma_0$  and the diagram of dual cones associated to the faces of  $\Gamma_0$  are pictured in Figures 1 and 2, respectively. We denoted by  $V_1$ ,  $V_2$  and  $V_3$  the vertices of  $\Gamma_0$  and by  $\gamma_{12}$  and  $\gamma_{23}$  its compact faces. One easily verifies that  $f$  is non degenerate with respect to both  $\Gamma_0$  and  $\Gamma_{gl}$ . Theorem 1.6(i) yields

$$\begin{aligned} Z_{\text{top},0}^{(1)}(s) &= J_{V_1}(s) + J_{V_2}(s) + J_{V_3}(s) \\ &\quad + \frac{s}{s+1}((-1) \text{Vol}(\gamma_{12})J_{\gamma_{12}}(s) + (-1) \text{Vol}(\gamma_{23})J_{\gamma_{23}}(s)) \\ &= 2 \frac{1}{(s+1)(6s+3)} + \frac{3}{(6s+3)^2} + \frac{s}{s+1} (2(-1) \cdot 1 \cdot \frac{1}{(6s+3)}) \\ &= \frac{-4s^2 + 3s + 3}{3(s+1)(2s+1)^2}. \end{aligned}$$

Alternatively we can construct the minimal embedded resolution of the germ of  $f^{-1}\{0\}$  at 0. It consists of 3 exceptional curves, intersecting as in Figure 3, where the dots correspond to intersections of the exceptional divisor with the strict transform of  $f^{-1}\{0\}$ . Since the numerical data of each component of this strict transform are  $(1, 1)$  we have by definition that

$$Z_{\text{top},0}^{(1)}(s) = 2 \frac{1}{6s+3} (-1 + 2 \frac{1}{s+1} + \frac{1}{4s+2}) + \frac{0}{4s+2},$$

yielding (fortunately) the same result. So  $-\frac{1}{2}$  is the only pole of order 2 and it is indeed the largest pole. Using Theorem 1.6(ii) or by considering the (global) embedded resolution of  $f^{-1}\{0\} \subset \mathbb{A}^2$  one can analogously compute that

$$Z_{\text{top}}^{(1)}(s) = \frac{72s^4 + 128s^3 + 77s^2 + 21s + 3}{3(s+1)^2(2s+1)^2},$$

which confirms Proposition 2.5.

*2.7. Remark.* Proposition 2.5 however is specific for non degenerate polynomials; it is not true for arbitrary  $f$ . One can easily construct counterexamples where  $f$  is not reduced, e.g.  $f = xy(x-1)^2(y-1)^2(x-2)^3(y-2)^3$  with double poles for  $Z_{\text{top}}^{(1)}(s)$  at  $-1$ ,  $-\frac{1}{2}$  and  $-\frac{1}{3}$ . An irreducible counterexample derived from it with the same double poles is  $f = xy(x-1)^2(y-1)^2(x-2)^3(y-2)^3 + (x-y)^7$ . ???

### 3. POLES OF MAXIMAL ORDER ARE OF THE FORM $-1/N$

(3.1) From the proofs in the previous section it was already clear that when  $f$  is non degenerate with respect to its Newton polyhedron, then a pole of order  $n$  of  $Z_{\text{top},0}(s)$  or  $Z_{\text{top}}(s)$  must be of the form  $-1/N$  with  $N \in \mathbb{N} \setminus \{0\}$ . We will prove this in general.

We now reconsider the defining expression of the topological zeta function in (0.1) in terms of the embedded resolution  $(X, h)$ . It is obvious that if  $\tilde{s}$  is a pole of order  $n$  of  $Z_{\text{top},0}(s)$  or  $Z_{\text{top}}(s)$ , then there exist  $n$  different  $E_i, i \in I \subset S$ , such that  $\cap_{i \in I} E_i \neq \emptyset$  and  $\tilde{s} = -\nu_i/N_i$  for all  $i \in I$ . The following result treats this situation in a slightly more general setting.

**3.2. Theorem.** *Let  $D = \sum_i N_i D_i$  be an effective divisor on a nonsingular variety  $Y$  of dimension  $n$ . Take an embedded resolution  $h : X \rightarrow Y$  of  $D$  in the sense of Hironaka's Main Theorem II [H, page 142], and let  $E_i, i \in S$ , be the irreducible components of  $h^{-1}(\text{supp } D)$ . Denote  $h^*D = \sum_{i \in S} N_i E_i$  and  $K_X = \sum_{i \in S} (\nu_i - 1)E_i + h^*K_Y$ . Suppose that there exist  $n$  different  $E_i, i \in I \subset S$ , such that  $\cap_{i \in I} E_i \neq \emptyset$  and  $\nu_i/N_i = t$  for all  $i \in I$ ; then  $t = 1/N$  for some  $N \in \mathbb{N} \setminus \{0\}$ .*

*Proof.* If at least one  $E_i, i \in I$ , is an irreducible component of the strict transform of  $D$ , then clearly  $t = 1/N_i$ . So from now on we suppose that all  $E_i, i \in I$ , are exceptional varieties.

At a certain step of the resolution process  $h$  one of the  $E_i, i \in I$ , is created as the exceptional variety of a blowing-up and the other ones are strict transforms of previously created exceptional varieties. We now consider the following situation (\*) of which this 'step' is a special case.

- Let  $\pi : X_1 \rightarrow X_0$  be a blowing-up of  $h$  with centre  $C_0$  of codimension  $d \geq 2$  in  $X_0$  and exceptional variety  $E_1 \subset X_1$ . Suppose that
- (i) there exists a point  $P \in E_1$  belonging to  $n$  different exceptional varieties of
  - (\*)  $h$ , say  $P \in E_1 \cap \tilde{E}_2 \cap \cdots \cap \tilde{E}_n$ , where  $\tilde{E}_j$  is the strict transform of  $E_j \subset X_0$  for  $j = 2, \dots, n$ ; and
  - (ii)  $\frac{\nu_1}{N_1 + a_1} = \frac{\nu_2}{N_2 + a_2} = \cdots = \frac{\nu_n}{N_n + a_n} = t$ , where the  $a_i \in \mathbb{Z}$  (and such that all  $N_i + a_i \neq 0$ ).

We claim that we may suppose (after renumbering) that  $E_2, \dots, E_d \supset C_0$  and  $E_{d+1}, \dots, E_n \not\supset C_0$ . Indeed since  $C_0$  has normal crossings with  $\cup_{\ell=2}^n E_\ell$  we can take local parameters  $y_1, \dots, y_n$  at  $Q = \pi(P)$  such that  $C_0$  is given locally at  $Q$  by  $y_1 = \dots = y_d = 0$  and the  $E_\ell, 2 \leq \ell \leq d$ , by some  $y_j = 0$ . Certainly at most  $d$  of the  $E_\ell$  can contain  $C_0$ , but since  $P \in \tilde{E}_\ell$  for all  $\ell$  in fact at most  $d - 1$  of them can satisfy  $E_\ell \supset C_0$ . On the other hand it is also clear that at most  $n - d$  of the  $E_\ell$  can satisfy  $E_\ell \not\supset C_0$ . This proves the claim.

So we are left with two possibilities :

- (1) no other exceptional variety of  $h$  contains  $C_0$ , or (say)
- (2) also  $E_{n+1} \supset C_0$ .

We will show that then

- (1')  $t = 1/N$  for some  $N \in \mathbb{N} \setminus \{0\}$ , and
- (2')  $t = \frac{\nu_{n+1}}{N_{n+1} + a_{n+1}}$  for some  $a_{n+1} \in \mathbb{Z}$ ,

respectively. Recall the well-known (and easily derived) equalities

$$N_1 = \sum_{i=2}^d N_i + \mu < +N_{n+1} > \quad \text{and} \quad \nu_1 = \sum_{i=2}^d (\nu_i - 1) + d < +\nu_{n+1} - 1 >,$$

where  $\mu$  is the multiplicity of the generic point of  $C_0$  on the strict transform of  $D$  on  $X_0$ , and in case (1) and (2) the terms within brackets do not and do occur, respectively. So

$$t = \frac{\nu_1}{N_1 + a_1} = \frac{\sum_{i=2}^d \nu_i + 1 < +\nu_{n+1} - 1 >}{\sum_{i=2}^d (N_i + a_i) + (\mu + a_1 - \sum_{i=2}^d a_i) < +N_{n+1} >},$$

which implies by the trivial Lemma 3.3 below that

$$t = \frac{1 < +\nu_{n+1} - 1 >}{\mu + a_1 - \sum_{i=2}^d a_i < +N_{n+1} >}.$$

This is what we claimed in (1') and (2').

Now we can prove the theorem by consecutive applications of our study of the situation (\*). Start with the blowing-up of  $h$  where  $\cap_{i \in I} E_i$  is created. In case (1) we are done. In case (2) we obtain (using the notation above)  $Q \in \cap_{j=2}^{n+1} E_j$  which induces by (2') a new situation (\*). We can now repeat the same arguments untill, by finiteness of the resolution, we encounter a case (1).  $\square$

**3.3. Lemma.** *Let  $k \geq 2$  and take  $b_i, c_i \in \mathbb{N} \setminus \{0\}$  for  $i = 1, \dots, k + 1$ . If  $\frac{b_1}{c_1} = \dots = \frac{b_k}{c_k}$  and  $\frac{b_1}{c_1} = \frac{b_2 + \dots + b_k + b_{k+1}}{c_2 + \dots + c_k + c_{k+1}}$ , then  $\frac{b_1}{c_1} = \frac{b_{k+1}}{c_{k+1}}$ .*

**3.4. Corollary.** *Any pole of order  $n$  of  $Z_{\text{top},0}(s)$  or  $Z_{\text{top}}(s)$  is of the form  $-1/N$  with  $N \in \mathbb{N} \setminus \{0\}$ .*

**3.5. Remark.** Theorem 3.2 also implies that any (complex) pole of order  $n$  of Igusa's local zeta function, associated to a polynomial in  $n$  variables over a  $p$ -adic field, has real part  $-1/N$  with  $N \in \mathbb{N} \setminus \{0\}$ . See for example [I,D] for this concept. An analogous result follows for the motivic Igusa zeta functions of [DL2] when one defines in a natural way a pole and its order for these zeta functions.

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