

DETERMINATION OF THE POLES OF THE TOPOLOGICAL ZETA FUNCTION FOR CURVES

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INTRODUCTION

(0.1) Let $f \in \mathbb{C}[x_1, \dots, x_n]$ and fix an embedded resolution (with normal crossings) $h : X \rightarrow \mathbb{A}^n$ of $f^{-1}\{0\}$. We denote by $E_i, i \in T$, the (reduced) irreducible components of $h^{-1}(f^{-1}\{0\})$, and by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor of respectively $f \circ h$ and $h^*(dx_1 \wedge \dots \wedge dx_n)$ on X . The $(N_i, \nu_i), i \in T$, are called the *numerical data* of the resolution (X, h) . For $I \subset T$ denote also $E_I := \cap_{i \in I} E_i$ and $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$.

Definition. To f one associates the topological zeta functions

$$Z_{\text{top}}(s) = \sum_{I \subset T} \chi(\overset{\circ}{E}_I) \prod_{i \in I} \frac{1}{\nu_i + sN_i}$$

and

$$Z_{\text{top},0}(s) = \sum_{I \subset T} \chi(\overset{\circ}{E}_I \cap h^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

Here $\chi(\cdot)$ denotes the (complex) Euler-Poincaré characteristic. Those zeta functions are invariants of respectively f and the germ of f at 0 and were introduced by Denef and Loeser in [DL]; the remarkable fact that the defining expressions do not depend on the chosen resolution is proved by expressing them as a limit of Igusa's local zeta functions. See also §1.

(0.2) Each component $E_i, i \in T$, induces a candidate pole $s_0 = -\frac{\nu_i}{N_i}$ for the topological zeta function of f . Now it is striking that 'most' candidate poles are actually bad. This fact would be elucidated if the following monodromy conjecture is true.

1991 *Mathematics Subject Classification.* 14B05 14H20 32S45 (11S40).

Key words and phrases. Topological zeta function, curve singularities, resolution graph.

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Conjecture [DL, (3.3.2)]. If s_0 is a pole of $Z_{\text{top},0}(s)$, then $e^{2\pi i s_0}$ is an eigenvalue of the local monodromy of f at some point of (the germ of) $f^{-1}\{0\}$.

See for example [L1,L2,V4] for some results. Assuming this conjecture we can remove the candidate poles eliminated by it; but still then very little is known about the remaining candidate poles.

(0.3) Suppose now that $n = 2$. Remark that in this case the topological zeta function of f can be defined unambiguously (without referring to Igusa's local zeta function), using the *canonical* embedded resolution of $f^{-1}\{0\}$.

Take an exceptional curve E_i intersecting exactly one or two times other components $E_j, j \in T$, and such that $\frac{\nu_j}{N_j} \neq \frac{\nu_i}{N_i}$ for these intersecting components. Then certain relations between the occurring numerical data immediately imply that the contribution of E_i to the residue of $-\frac{\nu_i}{N_i}$ for $Z_{\text{top}}(s)$ is zero; see §2. In a *generic* situation one can show that this statement is equivalent to the monodromy conjecture.

(0.4) In this paper we exactly determine all poles of $Z_{\text{top},0}(s)$ for $n = 2$ and any $f \in \mathbb{C}[x_1, x_2]$. In fact when (X, h) is the *canonical* embedded resolution of the germ of f at 0 we prove the following.

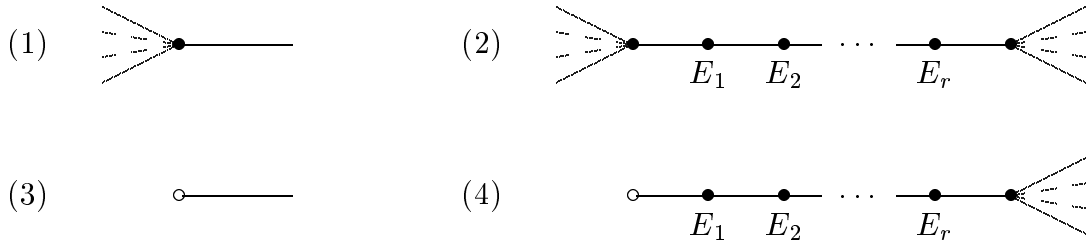
Theorem. $Z_{\text{top},0}(s)$ has at most one pole of order 2. Moreover s_0 is a pole of order 2 if and only if there exist two intersecting components E_i and E_j with $s_0 = -\frac{\nu_i}{N_i} = -\frac{\nu_j}{N_j}$, and in that case s_0 is the pole closest to the origin.

Theorem. We have that s_0 is a pole of $Z_{\text{top},0}(s)$ if and only if $s_0 = -\frac{\nu_i}{N_i}$ for some exceptional curve E_i intersecting at least 3 times other components or $s_0 = -\frac{1}{N_i}$ for some irreducible component E_i of the strict transform of f .

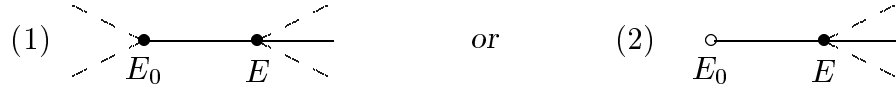
(In [V1] and [V3] we proved the analog of the last theorem for Igusa's local zeta function. It does not imply the present result for the topological zeta function, see §1.)

(0.5) Our proofs rely on the following new geometrical result which makes the resolution graph of the germ of f into an 'ordered tree'. In the (dual) resolution graph we associate to each exceptional curve and to each *analytically irreducible* component of the strict transform a vertex (represented respectively by a dot and a circle), and to each intersection between components E_i an edge, connecting the corresponding vertices. Also to each vertex $E_i, i \in T$, we associate the ratio $\frac{\nu_i}{N_i}$.

Theorem. (i) The $E_j, j \in T$, for which $\frac{\nu_j}{N_j} = \min_{i \in T} \frac{\nu_i}{N_i}$, together with their edges, form a connected part \mathcal{M} of the resolution graph. More precisely \mathcal{M} has one of the following forms (with $r \geq 0$) :



(ii) If for some exceptional curve E we have



in the resolution graph with $\frac{\nu_0}{N_0} < \frac{\nu}{N}$, then necessarily $\frac{\nu}{N} < \frac{\nu_i}{N_i}$ for all other components E_i that intersect E .

(iii) Starting from an end vertex of the minimal part \mathcal{M} , the numbers $\frac{\nu_i}{N_i}$ strictly increase along any path in the tree (away from \mathcal{M}).

Remark. When f is reduced the cases (3) and (4) of (i) (and case (2) of (ii)) cannot occur.

(0.6) In §1 we define Igusa's local zeta function and compare it with the topological zeta function. This section is not necessary to understand the results on $Z_{\text{top},0}(s)$, but provides some motivation and background. We first study the contribution of *one* exceptional curve E_i to the residue of $-\frac{\nu_i}{N_i}$ for $Z_{\text{top},0}(s)$ in §2. Then in §3 we prove the ordered tree structure of the resolution graph and we study some special cases. In §4 we determine all poles of $Z_{\text{top},0}(s)$ and we conclude with some related remarks and results on $Z_{\text{top}}(s)$.

1. IGUSA VERSUS TOPOLOGICAL

(1.1) Let K be a finite extension of the field \mathbb{Q}_p of p -adic numbers, R the valuation ring of K , P the maximal ideal of R , and $\bar{K} = R/P$ the residue field with cardinality q . For $z \in K$ we denote by $|z|$ its absolute value. To $f(x) \in K[x]$, $x = (x_1, \dots, x_n)$, one associates *Igusa's local zeta function*

$$Z_K(s) = \int_{R^n} |f(x)|^s |dx| \quad \text{and} \quad Z_{K,0}(s) = \int_{P^n} |f(x)|^s |dx|$$

for $\text{Re } s > 0$, where $|dx|$ denotes the Haar measure on K^n , normalized such that R^n has measure 1. Igusa [I1] showed that it is a rational function of q^{-s} , so it extends to a meromorphic function on \mathbb{C} . See [D2] for an overview of the work on this subject.

(1.2) Suppose now that f is defined over some number field F ; so we can define Igusa's local zeta function of f with respect to any completion K of F . For simplicity choose the resolution (X, h) of (0.1) to be defined over F (scheme-theoretically). There is a formula for Igusa's local zeta function in terms of (X, h) similar to the defining expression of the topological zeta function.

Theorem [D1, Theorem 3.1]. *Denote the reduction mod P of h and \mathring{E}_i , $I \in T$, respectively by $h_{\bar{K}}$ and $(\mathring{E}_I)_{\bar{K}}$. For almost all completions K of F (i.e. for all except a finite number) we have*

$$Z_K(s) = q^{-n} \sum_{I \subset T} C_I(\bar{K}) \prod_{i \in I} \frac{q-1}{q^{\nu_i + sN_i} - 1}$$

and

$$Z_{K,0}(s) = q^{-n} \sum_{I \subset T} C_{I,0}(\bar{K}) \prod_{i \in I} \frac{q-1}{q^{\nu_i+sN_i}-1},$$

where $C_I(\bar{K})$ and $C_{I,0}(\bar{K})$ are the number of \bar{K} -rational points on respectively $(\mathring{E}_I)_{\bar{K}}$ and $(\mathring{E}_I)_{\bar{K}} \cap h_{\bar{K}}^{-1}\{0\}$.

(1.3) Heuristically the topological zeta function is obtained as a limit when q tends to 1 of a series of Igusa's local zeta functions. Denef and Loeser give in [DL] an exact meaning to this argument (using algebraic approximation and ℓ -adic interpolation). As a corollary to their proof of this, they also obtain the following result, which relates poles of the topological zeta function to poles of Igusa's local zeta function.

Theorem [DL, Theorem 2.2]. *Let $f \in F[x_1, \dots, x_n]$ for some number field F . When s_0 is a pole of $Z_{\text{top}}(s)$ (respectively $Z_{\text{top},0}(s)$), then for almost all completions K of F , there exist infinitely many unramified extensions L of K such that s_0 is a pole of $Z_L(s)$ (respectively $Z_{L,0}(s)$).*

(1.4) So roughly a pole of the topological zeta function induces a pole of Igusa's local zeta function. (Intuitively this is already clear by the limit argument.) Now it is still an open question if conversely a pole of $Z_{K,0}(s)$ always induces a pole of $Z_{\text{top},0}(s)$. In fact the results of this paper yield an affirmative answer to this question when $n = 2$, since we will prove for $Z_{\text{top},0}(s)$ the analog of the following result.

Theorem ([V1, Corollary 3.3] and [V3, Theorem III.4.1]). *Let $f \in F[x_1, x_2]$ for some (big enough) number field F and suppose that (X, h) is the canonical embedded resolution of the germ of f at 0. Then for almost all completions K of F we have that s_0 is a pole of $Z_{K,0}(s)$ if and only if $s_0 = -\frac{\nu_i}{N_i}$ for some exceptional curve E_i that intersects at least 3 times other $E_j, j \in T$, or $s_0 = -\frac{1}{N_i}$ for some irreducible component of the strict transform of f .*

2. CONTRIBUTION OF ONE EXCEPTIONAL CURVE

(2.1) From now on we suppose that $n = 2$ and we fix $f \in \mathbb{C}[x, y]$. Let (X, h) be the canonical embedded resolution (with normal crossings) of the germ of $f^{-1}\{0\}$ at 0. So in particular h is a finite succession of blowing-ups, and no 'unnecessary' blowing-ups occur. We denote by $E_i, i \in T = T_e \cup T_s$, the (reduced) irreducible components of $h^{-1}(f^{-1}\{0\})$, where E_i is an exceptional curve for $i \in T_e$ and an irreducible component of the strict transform of $f^{-1}\{0\}$ for $i \in T_s$. For each $i \in T$ let N_i and $\nu_i - 1$ be the multiplicities of E_i in the divisor of respectively $f \circ h$ and $h^*(dx \wedge dy)$ on X . We have $N_i, \nu_i \geq 1$ and if f is reduced, then $(N_i, \nu_i) = (1, 1)$ for $i \in T_s$.

(2.2) Denote $\overset{\circ}{E}_i := E_i \setminus \cup_{j \neq i} E_j$ for $i \in T_e$. Then the defining expression for $Z_{\text{top},0}(s)$ reduces to

$$Z_{\text{top},0}(s) = \sum_{j \in T_e} \frac{\chi(\overset{\circ}{E}_j)}{\nu_j + sN_j} + \sum_{\{i,j\} \subset T} \frac{\chi(E_i \cap E_j)}{(\nu_i + sN_i)(\nu_j + sN_j)}.$$

(2.3) Fix now one exceptional curve E , intersecting k times other components E_1, \dots, E_k . For all $i = 1, \dots, k$ suppose that $\frac{\nu_i}{N_i} \neq \frac{\nu}{N}$ and set $\alpha_i = \nu_i - \frac{\nu}{N}N_i$. Then the contribution of E to the residue of $- \frac{\nu_i}{N_i}$ for $Z_{\text{top},0}(s)$ is

$$\mathcal{R} := \frac{1}{N}(2 - k + \sum_{i=1}^k \frac{1}{\alpha_i}).$$

We will prove in Proposition 2.8 that \mathcal{R} cannot be zero if $k \geq 3$. Therefore we need the following relations and inequalities for the α_i .

2.4. Theorem [L1, Lemme II.2]. *Let the exceptional curve E intersect k times other components E_1, \dots, E_k and set $\alpha_i = \nu_i - \frac{\nu}{N}N_i$ for $i = 1, \dots, k$. Then*

$$\sum_{i=1}^k \alpha_i = k - 2.$$

(For a short conceptual proof and generalizations, see [V2].)

2.5. Remark. Theorem 2.4 immediately implies what we claimed in (0.3), i.e. that $\mathcal{R} = 0$ when E_j intersects exactly 1 or 2 times other components.

2.6. Proposition [L1, Proposition II.3.1]. *Let the exceptional curve E intersect k times other components E_1, \dots, E_k and set $\alpha_i = \nu_i - \frac{\nu}{N}N_i$ for $i = 1, \dots, k$. For all $i = 1, \dots, k$ we have that $-1 \leq \alpha_i < 1$, equality occurring if and only if $k = 1$.*

2.7. Corollary.

- (i) *At most one E_i , $1 \leq i \leq k$, occurs such that $\alpha_i < 0$ ($\Leftrightarrow \frac{\nu_i}{N_i} < \frac{\nu}{N}$).*
- (ii) *If $k \geq 3$ then at most one E_i , $1 \leq i \leq k$, occurs such that $\alpha_i \leq 0$ ($\Leftrightarrow \frac{\nu_i}{N_i} \leq \frac{\nu}{N}$).*
- (iii) *If $k = 2$ then $\frac{\nu_1}{N_1} < \frac{\nu}{N} \Leftrightarrow \frac{\nu}{N} < \frac{\nu_2}{N_2}$.*

Proof. Almost trivial from 2.4 and 2.6; see also [V1, Corollary 2.2]. ■

2.8. Proposition. *Fix one exceptional curve E , intersecting $k \geq 3$ times other components E_1, \dots, E_k and such that $\frac{\nu_i}{N_i} \neq \frac{\nu}{N}$ for all $i = 1, \dots, k$. Then the contribution \mathcal{R} of E to the residue of $- \frac{\nu}{N}$ for $Z_{\text{top},0}(s)$ is never zero; more precisely*

- (i) $\mathcal{R} > 0 \Leftrightarrow \frac{\nu_i}{N_i} > \frac{\nu}{N}$ for all $i = 1, \dots, k$ and

(ii) $\mathcal{R} < 0 \Leftrightarrow \frac{\nu_i}{N_i} < \frac{\nu}{N}$ for some (and thus exactly one) $i \in \{1, \dots, k\}$.

Proof. Set $\alpha_i = \nu_i - \frac{\nu}{N}N_i$ for $i = 1, \dots, k$.

(i) In this case all α_i satisfy $0 < \alpha_i < 1$ (by Proposition 2.6), implying that $\mathcal{R} > \frac{2}{N}$.

(ii) Say that $\frac{\nu_k}{N_k} < \frac{\nu}{N}$; so $\alpha_k < 0$ and $0 < \alpha_i < 1$ for $i = 1, \dots, k-1$. By Theorem 2.4 we have that $-\alpha_k = \sum_{i=1}^{k-1} \alpha_i + 2 - k$ and thus

$$\mathcal{R} = \frac{1}{N}(2 - k + \sum_{i=1}^{k-1} \frac{1}{\alpha_i} - \frac{1}{\sum_{i=1}^{k-1} \alpha_i + 2 - k}).$$

Then it is clear that $\mathcal{R} < 0$ by Lemma 2.9 below. ■

2.9. Lemma. *Let $k \in \mathbb{N}, k \geq 3$, and $0 < \alpha_i < 1$ for $i = 1, \dots, k-1$ such that $0 < \sum_{i=1}^{k-1} \alpha_i + 2 - k$. Then*

$$2 - k + \sum_{i=1}^{k-1} \frac{1}{\alpha_i} - \frac{1}{\sum_{i=1}^{k-1} \alpha_i + 2 - k} < 0.$$

Proof. First it is easy to verify the following claim. Let $0 < a < 1$, $0 < b < 1$ and $0 < a + b - 1$. Then

$$(*) \quad -1 + \frac{1}{a} + \frac{1}{b} - \frac{1}{a + b - 1} < 0.$$

We now proceed by induction on k . The case $k = 3$ is just $(*)$. So take $k \geq 3$ and suppose that the lemma is true for k . We have to prove that

$$R := 1 - k + \sum_{i=1}^k \frac{1}{\alpha_i} - \frac{1}{\sum_{i=1}^k \alpha_i + 1 - k}$$

is strictly negative, assuming that

$$(1) \quad 0 < \alpha_i < 1 \text{ for } i = 1, \dots, k$$

and

$$(2) \quad 0 < \sum_{i=1}^k \alpha_i + 1 - k.$$

We decompose R as $R = R_1 + R_2$ where

$$R_1 := 2 - k + \sum_{i=1}^{k-1} \frac{1}{\alpha_i} - \frac{1}{\sum_{i=1}^{k-1} \alpha_i + 2 - k}$$

and

$$R_2 := -1 + \frac{1}{\alpha_k} + \frac{1}{\sum_{i=1}^{k-1} \alpha_i + 2 - k} - \frac{1}{\sum_{i=1}^k \alpha_i + 1 - k}.$$

Now from (1) and (2) we can derive

$$(3) \quad 0 < \sum_{i=1}^{k-1} \alpha_i + 2 - k < 1.$$

The induction hypothesis and the first inequality of (3) imply that $R_1 < 0$; furthermore since $0 < \alpha_k < 1$ and by (3) and (2), the fact that $R_2 < 0$ is just the claim (*). ■

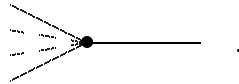
2.10. Remark. From Proposition 2.8 it is already clear that s_0 is a pole of $Z_{\text{top},0}(s)$ if $s_0 \in \{-\frac{\nu_i}{N_i} | i \in T_e\} \setminus \{-\frac{\nu_i}{N_i} | i \in T_s\}$, $s_0 = -\frac{\nu_j}{N_j}$ for *exactly one* $j \in T_e$, and E_j intersects at least 3 other components. Now both cases (i) and (ii) of Proposition 2.8 can occur for varying exceptional curves; so a priori it is not clear whether the contributions to the residue of different exceptional curves E_j with the same $\frac{\nu_j}{N_j}$ can add to zero. The ‘ordered tree’ structure of the resolution graph, determined in the next section, will imply that such a cancellation cannot happen, for we will show that case (i) of Proposition 2.8 occurs for *at most one* exceptional curve.

3. THE ‘ORDERED TREE’ STRUCTURE OF THE RESOLUTION GRAPH

(3.1) We suppose that the germ of f at 0 does not already have normal crossings, i.e. that $(f, 0)$ is not analytically isomorphic to $(x^N, 0)$ or $(x^N y^{N'}, 0)$. (In these cases $Z_{\text{top},0}(s)$ is respectively $\frac{1}{1+Ns}$ and $\frac{1}{(1+Ns)(1+N's)}$.)

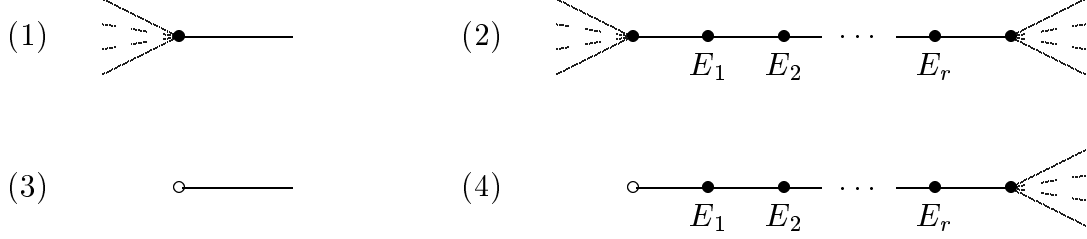
(3.2) In the (dual) embedded resolution graph of the germ of f at 0 one associates to each exceptional curve a vertex (represented by a dot) and to each intersection between exceptional curves an edge, connecting the corresponding vertices. Here we also associate to each *analytically irreducible* component of the strict transform a vertex (represented by a circle), and to its (unique !) intersection with an exceptional curve a corresponding edge. By the algorithm of embedded resolution it is clear that this graph is a (finite) tree with all circles end vertices.

Convention : we will picture a vertex with at least (e.g.) 3 edges as

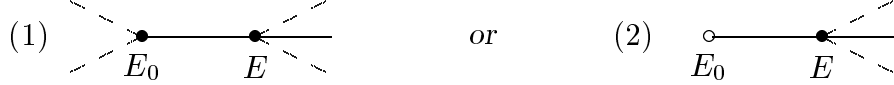


Now to each vertex $E_i, i \in T$, we associate the ratio $\frac{\nu_i}{N_i}$. The following theorem makes the resolution graph into an *ordered* tree with respect to the $\frac{\nu_i}{N_i}, i \in T$.

3.3. Theorem. (i) The $E_j, j \in T$, for which $\frac{\nu_j}{N_j} = \min_{i \in T} \frac{\nu_i}{N_i}$, together with their edges, form a connected part \mathcal{M} of the resolution graph. More precisely \mathcal{M} has one of the following forms (with $r \geq 0$) :



(ii) If for some exceptional curve E we have



in the resolution graph with $\frac{\nu_0}{N_0} < \frac{\nu}{N}$, then necessarily $\frac{\nu}{N} < \frac{\nu_i}{N_i}$ for all other components E_i that intersect E .

(iii) Starting from an end vertex of the minimal part \mathcal{M} , the numbers $\frac{\nu_i}{N_i}$ strictly increase along any path in the tree (away from \mathcal{M}).

Proof. Corollary 2.7 yields (ii), which clearly implies (iii) and the fact that \mathcal{M} is connected. We now prove the classification of (i) for \mathcal{M} .

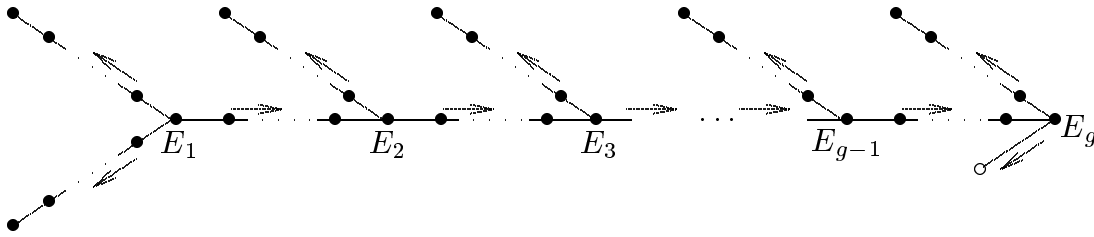
For $i \in T_e$ let k_i denote the number of intersections of E_i with other components. Take an exceptional curve E_i belonging to \mathcal{M} (if possible). By Theorem 2.4 we have that $k_i \neq 1$. If $k_i = 2$ then Corollary 2.7(iii) implies that the two intersecting components of E_i also belong to \mathcal{M} . Continuing the same argument yields that \mathcal{M} must contain a chain as in (2) or (4) with $r \geq 1$, and then by Corollary 2.7(ii) we have that \mathcal{M} must be exactly such a chain. Finally when \mathcal{M} does not contain any exceptional curve E_i with $k_i = 2$, then again Corollary 2.7(ii) implies that \mathcal{M} must be of the form (1) or (3), or (2) or (4) with $r = 0$. ■

Notation : Further in this section we still denote by \mathcal{M} the minimal part of Theorem 3.3.

3.4. Remark. (a) Suppose that f is reduced. Then the algorithm of embedded resolution easily implies that $\nu_i \leq N_i$ for $i \in T_e$ and that $\nu_i < N_i$ if E_i intersects the strict transform. Consequently the cases (3) and (4) of (i) (and case (2) of (ii)) cannot occur in Theorem 3.3. On the other hand in (i) there exist examples of case (1) and case (2) for any $r \geq 0$.

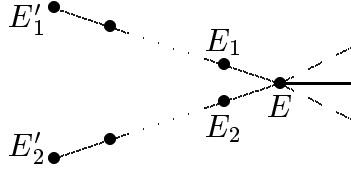
(b) Moreover there exist examples (with non-reduced f) of case (3) and case (4) for any $r \geq 0$. See also Proposition 3.8.

3.5. Example. When f is analytically irreducible at 0 (with g different Puiseux exponents), then the resolution graph has the following form.



Its minimal part \mathcal{M} consists just of E_1 and the ratio $\frac{\nu_i}{N_i}$ strictly increases in the sense of the arrows. This fact was already discovered by Strauss [S, Corollary 2.1] through complicated computations. It is also implied by the following more general result.

3.6. Proposition. *If the resolution graph contains a part of the form below, then the minimal part \mathcal{M} is E .*

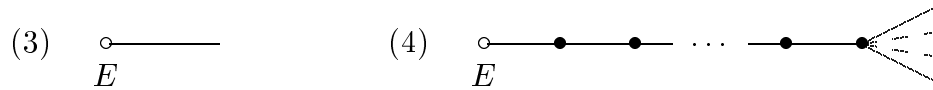


Proof. Let E intersect $E_1, E_2, \dots, E_k (k \geq 3)$ and set $\alpha_i = \nu_i - \frac{\nu}{N} N_i$ for $i = 1, \dots, k$. An appropriate combination of Theorem 2.4 for all components on the path connecting E_1 to E'_1 yields that $\alpha_1 = \frac{N'_1}{N}$ (see also [I1] or [L1]). Now for example by [L1, Lemme II.2] we have that $N'_1 | N$ and so $\alpha_1 \leq \frac{1}{2}$. The same argument yields $\alpha_2 \leq \frac{1}{2}$ and then again Theorem 2.4, now applied to E , implies that $\sum_{i=3}^k \alpha_i \geq k - 3$. Since for $i = 3, \dots, k$ we have that $\alpha_i < 1$ (Proposition 2.6) we must also have that $0 < \alpha_i$. ■

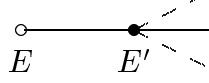
Remark. In particular this implies that the resolution graph can contain *at most one* part of the described form. (One can also see this by proving that in such a part E'_1 or E'_2 must be the first created exceptional curve in the resolution process.) So we obtain an important restriction on the shape of the (pure) resolution tree using our result on the ‘ordered tree’ structure with respect to the $\frac{\nu_i}{N_i}$.

(3.7) We already mentioned in Remark 3.4 that an analytically irreducible component of the strict transform cannot be part of the minimal set \mathcal{M} if f is reduced. In fact the cases (3) and (4) of Theorem 3.3(i) are quite rare; they are only possible if some irreducible component of $f^{-1}\{0\}$ is smooth at 0 and occurs moreover with ‘high’ multiplicity in the divisor of f , as shown below.

3.8. Proposition. *Let $f = \prod_{i \in I} f_i^{N_i}$ be the decomposition of f in irreducible factors, and denote by μ_i the multiplicity of f_i at 0. If the minimal set \mathcal{M} in Theorem 3.3(i) is as in case (3) (respectively case (4)), then there exists $j \in I$ such that E is the strict transform of $f_j^{-1}\{0\}$ and such that $\mu_j = 1$ and $N_j > \sum_{i \neq j} \mu_i N_i$ (respectively $N_j \geq \sum_{i \neq j} \mu_i N_i$).*

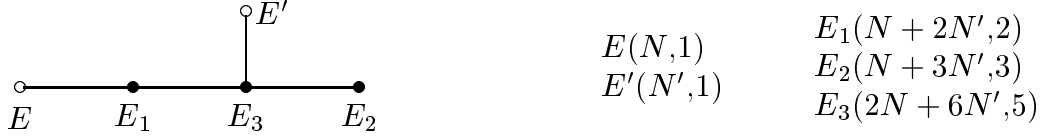


Proof. A priori it is not clear that E is the strict transform of some $f_i^{-1}\{0\}$, it could be an analytically irreducible component of it. So let E be an analytically irreducible component of the strict transform of $f_j^{-1}\{0\}$, and denote by E' the exceptional curve intersecting E .



Let E_1 be the first created exceptional curve in the resolution process; it has numerical data $(N_1, \nu_1) = (\sum_{i \in I} \mu_i N_i, 2)$. Using the algorithm of embedded resolution it is not difficult to show that if $\frac{\nu_1}{N_1} \leq \frac{1}{N_j}$, then also $\frac{\nu'}{N'} \leq \frac{1}{N_j}$ (respectively if $\frac{\nu_1}{N_1} < \frac{1}{N_j}$, then also $\frac{\nu'}{N'} < \frac{1}{N_j}$). Now in case (3) we have that $\frac{1}{N_j} < \frac{\nu'}{N'}$, and thus $\frac{1}{N_j} < \frac{\nu_1}{N_1}$, which is equivalent to $\sum_{i \neq j} \mu_i N_i < (2 - \mu_j)N_j$. Clearly this is only possible if $\mu_j = 1$ (which implies that $f_j^{-1}\{0\}$ is analytically irreducible) and consequently $N_j > \sum_{i \neq j} \mu_i N_i$. Case (4) is analogous. ■

3.9. Example. Take $f = x^N(y^2 - x^3)^{N'}$. Let E_1, E_2 and E_3 denote the exceptional curves (ordered as created), and E and E' the strict transforms of respectively $\{x = 0\}$ and $\{y^2 - x^3 = 0\}$. The resolution graph and numerical data are as follows.



As N and N' vary, cases (1), (3) and (4) of Theorem 3.3(i) can occur :

case (1) $\Leftrightarrow \mathcal{M}$ is $\Leftrightarrow N < 2N'$,

case (4) $\Leftrightarrow \mathcal{M}$ is $\Leftrightarrow N = 2N'$,

case (3) $\Leftrightarrow \mathcal{M}$ is $\Leftrightarrow N > 2N'$.

(3.10) Let $\prod_{i=1}^k (y - \lambda_i x)^{a_i}$ be the decomposition in linear factors of the lowest degree term of f (so $k \geq 1$ and all λ_i are different). Let also E denote the first created exceptional curve in the resolution process; it has numerical data $(N, \nu) = (\sum_{i=1}^k a_i, 2)$. We will prove that if $k \geq 3$ and if the a_i differ ‘not to much’, then the minimal set \mathcal{M} is just E .

At the stage of the process when E is just created it intersects exactly k times the strict transform of $f^{-1}\{0\}$ in points P_1, \dots, P_k corresponding to the tangent directions in 0, determined by respectively $\{y - \lambda_1 x = 0\}, \dots, \{y - \lambda_k x = 0\}$. We leave the following fact as an exercise for the reader.

Lemma. For $i = 1, \dots, k$ let E_i denote the component in the resolution space X that intersects E in the point corresponding to P_i . Then there exists $\ell_i \in \mathbb{N}$ such that $N_i = \ell_i N + a_i$ and $\nu_i = \ell_i \nu + 1$.

3.11. Proposition. Let $\prod_{i=1}^k (y - \lambda_i x)^{a_i}$ be the decomposition in linear factors of the lowest degree term of f and let E denote the first created exceptional curve in the resolution process.

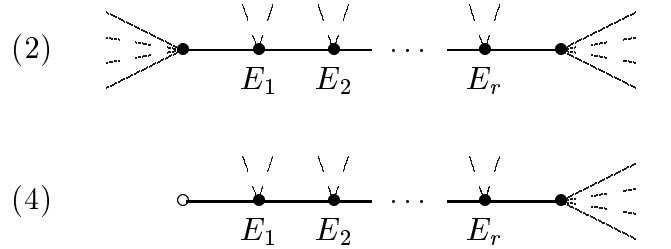
(i) Suppose that $k \geq 3$. Then \mathcal{M} is E if and only if for all $i = 1, \dots, k$ we have that $a_i < \sum_{\ell \neq i} a_\ell$. Also \mathcal{M} strictly contains E if and only if $a_i = \sum_{\ell \neq i} a_\ell$ for some (and thus exactly one) $i \in \{1, \dots, k\}$.

(ii) Suppose that $k = 2$. Then \mathcal{M} contains E if and only if $a_1 = a_2$.

Proof. Using the notations of (3.10), we have for $i = 1, \dots, k$ that the inequality $\frac{\nu}{N} < \frac{\nu_i}{N_i}$ is equivalent to $\frac{\nu}{N} < \frac{1}{a_i}$ (by the lemma) and thus also to $a_i < \sum_{\ell \neq i} a_\ell$. Analogously $\frac{\nu}{N} = \frac{\nu_i}{N_i} \Leftrightarrow a_i = \sum_{\ell \neq i} a_\ell$. All statements follow now easily. \blacksquare

Remark. In Example 3.9 the case $N = 2N'$ is an example of Proposition 3.11 (ii).

(3.12) To conclude this section we mention that Theorem 3.3 can easily be generalized for an *arbitrary* embedded resolution of the germ of f at 0. The only difference for an arbitrary resolution is that the cases (2) and (4) of (i) must be extended to



i.e. the ‘connecting’ exceptional curves E_1, \dots, E_r can intersect more than 2 other components. The statements (ii) and (iii) remain valid.

4. POLES OF THE TOPOLOGICAL ZETA FUNCTION

(4.1) We are now ready to determine all poles of $Z_{\text{top},0}(s)$, using the ‘residue contribution’ result of Proposition 2.8 and the ordered tree structure of the resolution graph. First Theorem 3.3 immediately implies that there is at most one pole of order two.

4.2. Theorem. $Z_{\text{top},0}(s)$ has at most one pole of order 2. Moreover s_0 is a pole of order 2 if and only if there exist two intersecting components E_i and E_j with $s_0 = -\frac{\nu_i}{N_i} = -\frac{\nu_j}{N_j}$, and in that case s_0 is the pole closest to the origin.

Such a pole occurs if and only if we have the case (2) or (4) in Theorem 3.3(i). Moreover when f is reduced only case (2) is possible.

4.3. Theorem. *We have that s_0 is a pole of $Z_{\text{top},0}(s)$ if and only if $s_0 = -\frac{\nu_i}{N_i}$ for some exceptional curve E_i intersecting at least 3 times other components or $s_0 = -\frac{1}{N_i}$ for some irreducible component E_i of the strict transform of f .*

Proof. Clearly s_0 can only be a pole if one of these conditions is satisfied (even if s_0 has order 2). Conversely let now $s_0 = -\frac{\nu_i}{N_i}$ exactly for $i \in I \subset T$. Two possibilities occur.

(I) The $E_i, i \in I$, form the minimal part \mathcal{M} of Theorem 3.3(i). If \mathcal{M} has the form (1) or (3), respectively (2) or (4), then s_0 is a pole of order 1, respectively 2. For case (1) this is implied by Proposition 2.8(i), the other cases are trivial.

(II) No $E_i, i \in I$, belongs to \mathcal{M} . By Remark 2.5 we have that $E_i, i \in I$, can only contribute to the residue of s_0 if $i \in T_s$ or if $i \in T_e$ and E_i intersects at least 3 other components. Now by respectively Theorem 3.3(iii) and Proposition 2.8(ii) all such contributions are strictly negative. Since by assumption there is at least one contribution we are done. ■

Remark. Theorems 4.2 and 4.3 are trivially true for the cases excluded in (3.1).

(4.4) More generally Denef and Loeser [DL] associate to $r \in \mathbb{N} \setminus \{0\}$ and $f \in \mathbb{C}[x, y]$ the topological zeta function

$$Z_{\text{top},0}^{(r)}(s) = \sum_{\substack{j \in T_e \\ r|N_j}} \frac{\chi(\overset{\circ}{E}_j)}{\nu_j + sN_j} + \sum_{\substack{\{i,j\} \subset T \\ r|N_i, r|N_j}} \frac{\chi(E_i \cap E_j)}{(\nu_i + sN_i)(\nu_j + sN_j)}.$$

(Of course this is possible in any dimension and also globally.) This expression is obtained as a limit of Igusa's local zeta functions, twisted by a character of order r , see [DL].

The analog of Theorem 4.2 remains true for $Z_{\text{top},0}^{(r)}(s)$ but we cannot expect results as in Theorem 4.3. To see this fix one exceptional curve E with $r|N$, intersecting $k \geq 3$ times other components E_1, \dots, E_k . For all $i = 1, \dots, k$ suppose that $\frac{\nu_i}{N_i} \neq \frac{\nu}{N}$ and set $\alpha_i = \nu_i - \frac{\nu}{N}N_i$. The contribution $\mathcal{R}^{(r)}$ of E to the residue of $s_0 = -\frac{\nu}{N}$ for $Z_{\text{top},0}^{(r)}(s)$ is

$$\mathcal{R}^{(r)} = \frac{1}{N}(2 - k + \sum_{\substack{i=1 \\ r|N_i}}^k \frac{1}{\alpha_i}).$$

The fact that not necessarily all $i = 1, \dots, k$ occur in the sum above yields that $\mathcal{R}^{(r)}$ can sometimes be zero.

An easy example is $f = x^{N_1}y^{N_2}(x-y)^{N_3}(x+y)^{N_4}$ where $3N_1 = N_2 + N_3 + N_4$, $r|N_1$ and $r \nmid N_i$ for $i = 2, 3, 4$. Its unique exceptional curve has numerical data $(4N_1, 2)$, intersects the 4 other components, but does not induce a pole of $Z_{\text{top},0}^{(r)}(s)$; in fact $Z_{\text{top},0}^{(r)}(s)$ is identically zero.

(4.5) Suppose that f is defined over some number field F . Then for almost all completions K of F we have that Theorem 3.3 immediately implies the analog of Theorem 4.2 for

Igusa's local zeta function $Z_{K,0}(s)$. So $Z_{K,0}(s)$ has at most one pole of order 2 and if so, this pole of order 2 is the pole closest to the origin. This is a new result, refining [L1, Lemme IV.2.3] and [V3, III4].

(4.6) We conclude with some remarks on the ‘global’ zeta function $Z_{\text{top}}(s)$. We suppose now that (X, h) is the canonical embedded resolution of $f^{-1}\{0\}$ in \mathbb{A}^2 and keep using all other notations of (2.1). Now $f^{-1}\{0\}$ can have several singular points, which all contribute to the zeta function.

(i) One easily sees that Theorem 4.2 cannot be true for $Z_{\text{top}}(s)$; of course it is still true that s_0 is a pole of order 2 if and only if there exist two intersecting components E_i and E_j with $s_0 = -\frac{\nu_i}{N_i} = -\frac{\nu_j}{N_j}$, but an arbitrary number of poles of order 2 can occur.

(ii) Also Theorem 4.3 is in general not true for $Z_{\text{top}}(s)$. To find a counterexample one can search for a curve $f^{-1}\{0\}$ with (at least 2) singularities P_i , each producing in its local resolution an exceptional curve E_i (intersecting at least 3 times other components), such that all E_i have the same ratio of numerical data $\frac{\nu_i}{N_i}$, and such that all contributions R_i to the residue of $s_0 = -\frac{\nu_i}{N_i}$ for $Z_{\text{top}}(s)$ satisfy $\sum_i R_i = 0$. A concrete example with moreover f irreducible is

$$f = [y^3(y^2 - x^2) + x^6] \cdot [y^3(y^2 - (x - 1)^2) + (x - 1)^6] \cdot \prod_{i=1}^5 (y - \lambda_i(x + 1)) + y^N$$

where N is big enough. It has 3 singular points : an ordinary 5-fold singularity and two times the germ of $\{y^3(y^2 - x^2) + x^6 = 0\}$ at 0; their contribution to the residue of $s_0 = -\frac{2}{5}$ for $Z_{\text{top}}(s)$ is respectively $\frac{16}{15}$ and two times $-\frac{8}{15}$.

(iii) When f is reduced it is still true that $s_0 = -1$ (induced by the strict transform of $f^{-1}\{0\}$) is always a pole of $Z_{\text{top}}(s)$, since in this case all contributions to its residue are strictly negative (see Remark 3.4).

(4.7) We can give a negative answer to the analogous question in (1.4) for $Z_{\text{top}}(s)$ and $Z_K(s)$. I.e. if f is defined over a number field F then it can happen that s_0 is a pole of $Z_K(s)$ for almost all completions K of F , but that it is not a pole of $Z_{\text{top}}(s)$. This follows from (4.6(ii)) and [V1, Theorem 3.2], where we proved that an exceptional curve E_i , intersecting at least 3 times other components, always induces a pole of $Z_K(s)$ for almost all completions K of a big enough F .

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